Structural System in Architecture
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Lecture - 16
Deflection of Beams

Welcome to the online NPTEL certification course on Structural Systems in Architecture. Today, we will start with the lectures of week 4 i.e.,the module number 4. The module number 4 will be of the Frame Structure Analysis and Design and in addition we'll also discuss about Deflection of Beams.

Concepts Covered

The following concepts are covered in this lecture:

- Equation of Deflection
- Boundary Conditions
- Double Integration Method

Learning Objectives

The learning objectives of this lecture are given below:

- Deducing the equation of beam deflection.
- Outlining the double integration method to evaluate slope and deflection of beams.
- Solving numerical examples on beam deflection.

Equation of Deflection

Before moving to the equation of deflection let us first recall something which we already know from our previous lectures. And that is, in a two dimensional coordinate plane, if I place a particular system or structural systems and if I applied some kind of a load and if the load is not axial but transverse kind of a load, it will cause the structure to bend.

Suppose we have a beam as given in the Figure 1. Now if this beam is subjected to transverse load across its length it will experience some bending moment. Then as a result, each cross-section of the beam will try to resist the local bending moment created in that particular area. Consequently, the portion will tilt accordingly resulting in the formation of a curvature in the concerned portion of the beam. This phenomenon will continue throughout the length of the beam resulting in the overall bending of the beam. Nonetheless, the curvature will not be the same throughout.

Figure 1 Bending in a simply supported and a cantilevered beam

The relationship between this profile of curvature and the external moment can be given by the Moment- Curvature Relationship of the Bending Theory which we have already studied in the last week's lecture and is given below:

$$\frac{M}{I} = \frac{E}{R}$$

Now let us delve further and put that particular curvature in a two dimensional plane having x and y axes as shown in the Figure 2.

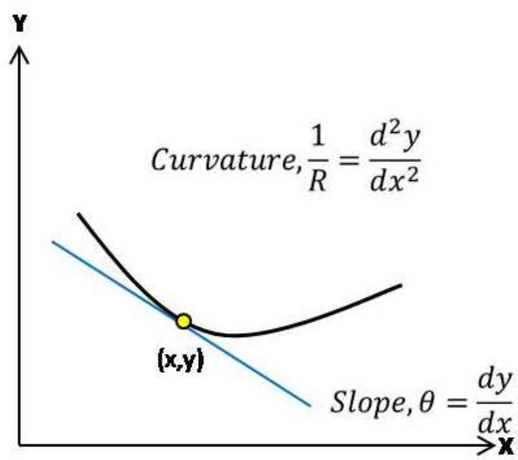


Figure 2 Beam curvature represented in a two-dimensional graph

But, we know that the curvature of any function is nothing but $\frac{d^2y}{dx^2}$, which is the second order of differentiation of the said function. So here we have, 2

$EI\left(\frac{1}{R}\right) = M$ [Moment-Curvature Relationship]

i.e., $EI\left(\frac{d^2y}{dx^2}\right) = M$

Integrating the above equation once we get,

$$EI\left(\frac{dy}{dx}\right) = \int M + C_1$$

This is nothing but the slope equation.

Next, integrating the above equation twice we get,

$$EIy = \iint M + \int C_1 + C_2$$

This is the equation of deflection. C_1 and C_2 are the integration constants whose values can be found out by appropriate boundary conditions. As a boundary value problem is a differential equation (or system of differential equations) to be solved in a domain on whose boundary a set of conditions is known. So in this case boundary conditions are those when you already know some of the values such as the deflection values in a particular point whose x and y coordinates are known to you. Then from there you can easily find out the values of the constants C1 and C2.

Only after finding the values of C_1 and C_2 this equation can be further solved and the slope, θ can be computed which is given by

Slope, $\theta = \frac{dy}{dx}$

Methods of Determining Slope and Deflection of a Beam

There are various types of methods of determining slope and deflection of a beam but these are applicable to different types of problems.

1. **Double Integration Method**

   This method is applicable for beams which are symmetrically loaded along with a continuity of moment throughout. In other words, the moment doesn't change along the length of the beam.

2. **Macaulay's Method**

This method is an improvised version of the double integration method and is applicable for complex kind of loading along with a discontinuity in its moment.

3. **Moment-Area Method**

   This is a semi-graphical method applicable in some specific cases. With this method where on one hand few very complex problems can be solved easily whereas on the other hand some simple problems cannot be solved that simply. However, it is important to note that the Moment-area Method is used for further development of other methods for complicated structures.

4. **Unit Load Method**

   The Unit Load Method is also used in some specific cases.

So out of the four methods, I must say that this Macaulay's method is little broader. So you can actually use it for the various types of beams. But you'll require some more lectures to understand this method. So I'll just skip it for this particular discussion. Thus, we will only study the double integration method for the time being and see how the deflections of the beams can be deduced from it.

Boundary Conditions

Now let us see the boundary condition, which is very important both in the double integration method and the Macaulay's method. As mentioned earlier, a boundary value problem is a system of ordinary differential equations with solution and derivative values specified at more than one point. Most commonly, the solution and derivatives are specified at just two points (the boundaries) defining a two-point boundary value problem.

So, in this case the boundary conditions are when there is some specific data or some specific condition that we already know from the given curvature or given slope of a particular structure. It depends upon the support systems most often. However, sometimes it may also depend upon the support and loading both and also the application of loading. To better understand this let us see the following examples.

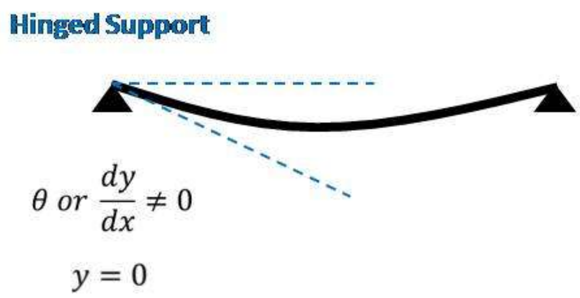


Figure 3 Example-1

Let us consider a beam here of length L consisting of hinged supports at its ends. Then, at the supports the moments will be 0. As a result, the beam will bend as shown in the Figure 3 by an angle, say θ. Clearly, the beam will not have any deflection at the supports; hence y=0.

Then, at x=0 and y=0, we have

$$\theta = \frac{dy}{dx} \neq 0$$

Also at x=L, y=0, we have

$$\theta = \frac{dy}{dx} \neq 0$$

Thus, we have achieved boundary conditions at both the supports.

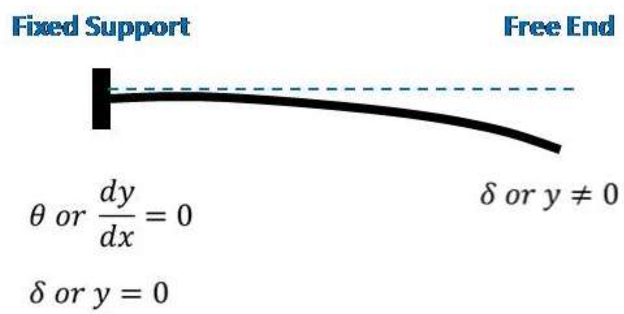


Figure 4 Example-2

Similarly, in the Figure 4 we have a cantilevered beam fixed at one end and free at another. The length of the beam is same as previous, i.e., L. Here at the fixed end, moment is not equal to zero and hence the support at this end will try to restrain the moment. This will eventually cause a deflection towards the other end of the beam.

So, at the fixed end we have,

x = 0, δ or y = 0

And, θ or $\frac{dy}{dx} = 0$

At the free end we have,

x = L but δ or y ≠ 0

Thus, here we cannot achieve the boundary conditions at the free end though we've achieved it on the fixed end. A beam must have at least two boundary conditions. After knowing the boundary conditions the constants C_1 and C_2 can be computed very easily.

Double Integration Method

Case-1(a): Cantilever Beam with Concentrated Load at the Free End

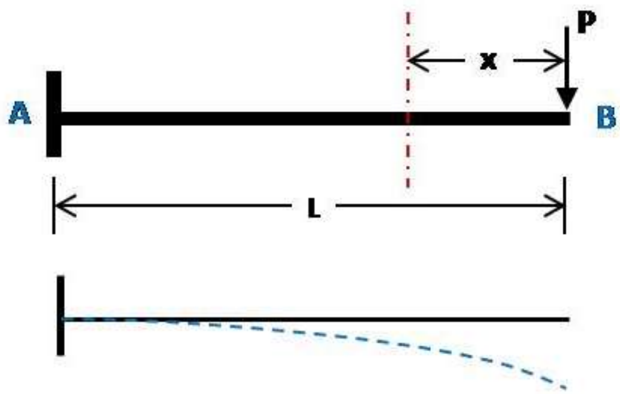


Figure 5 Cantilever beam with concentrated load

In the Figure 5 we have a cantilevered beam of length L fixed at A and free at B. A point load of P units is acting on the beam at B.

We know that,

$$EI\frac{d^2y}{dx^2} = M$$

Now,

Let us take a section at X, at a distance of x units from B.

Then, moment at X = Px

So we have,

$$EI\frac{d^2y}{dx^2} = Px$$

Integrating this equation once we get,

$EI\frac{dy}{dx} = \frac{P}{2}x^2 + C_1$ [Since $\int x = \frac{x^{n+1}}{n+1}$]

This is the slope equation of the given beam.

Integrating the equation twice (or integrating the above equation once) we get,

$EIy = \frac{P}{6}x^3 + C_1x + C_2$ [Deflection equation]

Next,

Let us consider the free-body diagram of the beam in a two-dimensional graph having x and y axes and origin at B.

Then, at x=L (i.e., at the point A)

$\theta = 0$ and δ or $y = 0$ [Boundary conditions]

Substituting these values in the slope equation we get,

$0 = \frac{P}{2}L^2 + C_1$

i.e., $C_1 = -\frac{PL^2}{2}$

Also, substituting the boundary conditions in the equation of deflection, we get

$0 = \frac{PL^3}{6} - \frac{PL^2}{2}L + C_2 = \frac{PL^3}{6} - \frac{PL^3}{2} + C_2 = -\frac{PL^3}{3} + C_2$

i.e., $C_2 = \frac{PL^3}{3}$

Thus, the slope equation can be written as

$EI\frac{dy}{dx} = \frac{P}{2}x^2 - \frac{PL^2}{2}$

So at x = 0,

$EI\frac{dy}{dx} = 0 - \frac{PL^2}{2} = -\frac{PL^2}{2}$

i.e., $\frac{dy}{dx} = -\frac{PL^2}{2EI} = \theta_B$

Again, the equation of deflection can be written as

$EIy = \frac{P}{6}x^3 - \frac{PL^2}{2}x + \frac{PL^3}{3}$

At x = 0,

$y = \frac{PL^3}{3EI} = \delta_B$

Case-1(b): Cantilever Beam with Concentrated Load anywhere on the Beam

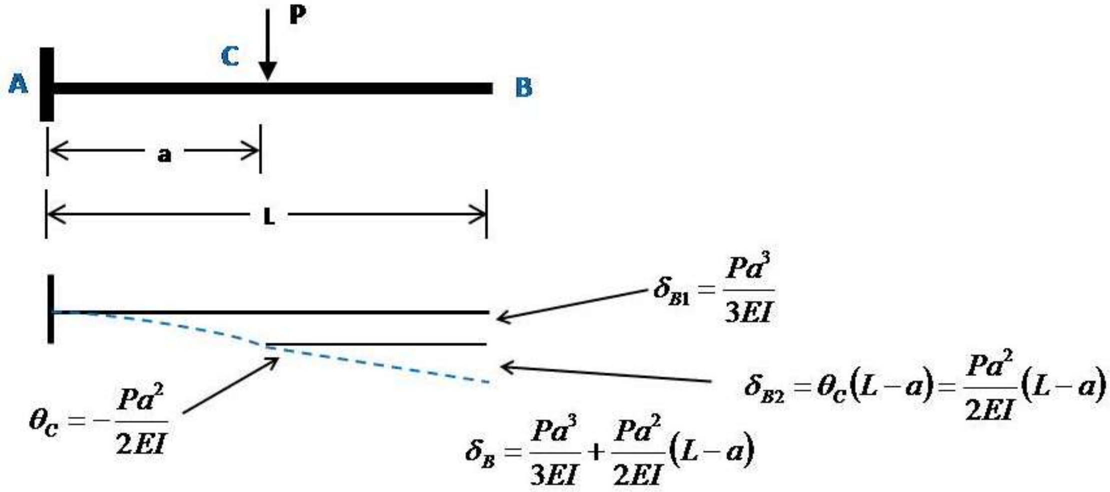


Figure 6 Cantilever beam with concentrated load at its centre

Similarly, if the load P is move anywhere at a distance "a" from A then the equations will be as follows:

Deflection due to the load at C (δ_{B1}) = $\frac{Pa^3}{3EI}$

And, $\theta_C = -\frac{Pa^2}{2EI}$

Now,

From this point onwards the deflection will be a straight line as there is no effect of any load between C and B, which can be given by

$$\delta_{B2} = \theta_c(L-a) = \frac{Pa^2}{2EI}(L-a)$$

Therefore,

The total deflection at B = $\delta_B = \delta_{B1} + \delta_{B2} = \frac{Pa^3}{3EI} + \frac{Pa^2}{2EI}(L-a)$

Case-2: Cantilever Beam with UDL

Here we have the cantilever beam AB subjected to a UDL of w KN/m as shown in the Figure 7.

Let us consider a section x-x' at x units from B.

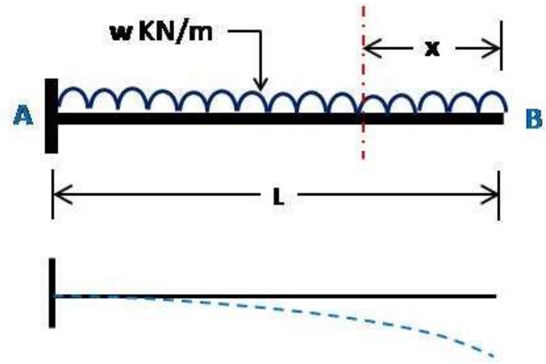


Figure 7 Cantilever beam subjected to UDL

So here we have,

$$EI\frac{d^2y}{dx^2} = M = \frac{w}{2}x^2$$

Integrating the above equation we get,

$$EI\frac{dy}{dx} = \frac{w}{6}x^3 + C_1 \text{ [Slope equation]}$$

Again integrating the above equation we get,

$$EIy = \frac{w}{24}x^4 + C_1x + C_2 \text{ [Deflection equation]}$$

Then we have,

Boundary Condition-1:

At A,

x = L and δ or y = 0

Substituting these values in the slope equation we get,

$$0 = \frac{w}{6}L^3 + C_1$$

$$\text{i.e., } C_1 = -\frac{wL^3}{6}$$

Also,

Boundary Condition-2:

At A, x = L and δ or y = 0

Substituting these values in the equation of deflection we get,

$$0 = \frac{wL^4}{24} - \frac{wL^4}{6} + C_2$$

$$\text{i.e., } C_2 = \frac{wL^4}{8}$$

Thus, we can re-write the slope equation as

$$EI\frac{dy}{dx} = \frac{w}{6}x^3 - \frac{wL^3}{6}$$

At the free end B, x = 0. So,

$$\theta_B = \frac{dy}{dx} = -\frac{wL^3}{6EI}$$

Next, we can re-write the equation of deflection as

$$EIy = \frac{w}{24}x^4 - \frac{PL^3}{6}x + \frac{wL^4}{8}$$

At free end,

$$\delta_B = \frac{wL^4}{8EI}$$

Case-3: Cantilever Beam

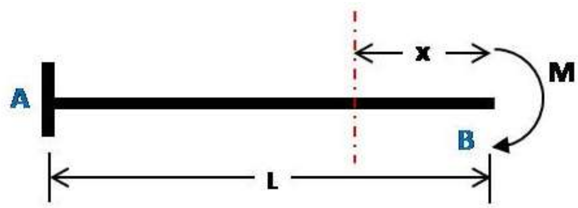


Figure 8 Cantilever beam

Now let us compute by taking moment in general for the given cantilever beam.

Then we have,

$$EI\frac{d^2y}{dx^2} = M$$

Integrating the above equation we get,

$$EI\frac{dy}{dx} = Mx + C_1 \text{ [Slope equation]}$$

Again integrating the above equation we get,

$$EIy = \frac{M}{2}x^2 + C_1x + C_2 \text{ [Deflection equation]}$$

Boundary condition-1:

At A, we have

x = L, slope = 0

So substituting the values in the slope equation we get,

$0 = ML + C_1$

i.e., $C_1 = -ML$

Boundary condition-2:

At A we have,

x = L, δ or y = 0

Substituting these values in the equation of deflection we get,

$$0 = \frac{M}{2}L^2 - ML^2 + C_2$$

i.e., $C_2 = \frac{ML^2}{2}$

Then, the equation of slope can be re-written as

$$EI\frac{dy}{dx} = Mx - ML$$

At the free end (x = 0)

$$\theta_B = \frac{dy}{dx} = -\frac{ML}{EI}$$

Also, the equation of deflection can be re-written as

$$EIy = \frac{M}{2}x^2 - MLx + \frac{ML^2}{2}$$

At the free end (x = 0)

$$\delta_B = y = \frac{ML^2}{2EI}$$

Example-3

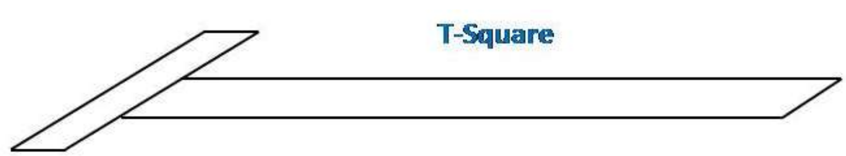


Figure 9 T-square

Length = 900mm

Width = 60mm

Thickness = 5mm

E = 50000 N/mm^2

Density = 4Kg/m^3

Now let us find the deflection of a T-square whose dimensions, Young's Modulus of elasticity and the density of material is already given.

We have,

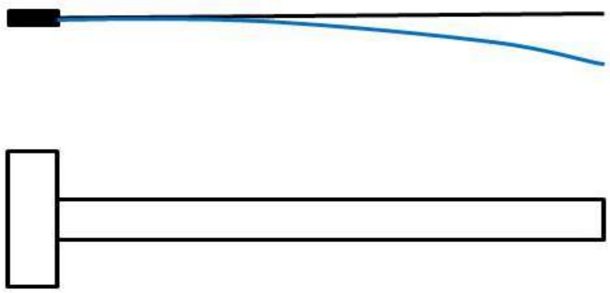
Self weight of the T-square (w) = (0.06 X 0.005 X 1) X 4 = 0.0012 Kg/m

= (0.0012 x 10) = 0.012 N/m

Now,

We know that the maximum deflection in a cantilever beam with UDL = $\frac{wL^4}{8EI}$

So,



$I = \frac{60\times5^3}{12} = 625\ mm^4$

$\delta_B = \frac{0.012\times900^4}{8\times50000\times625} = 31.5\ mm$

$I = \frac{5\times60^3}{12} = 90{,}000\ mm^4$

$\delta_B = \frac{0.012\times900^4}{8\times50000\times90000} = 0.22\ mm$

Clearly,

The deflection in both the cases is poles apart.

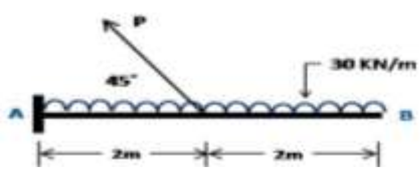
For this reason the beams are designed having greater depths so as to increase the value of I and therefore decrease the deflection as the former is inversely proportional to the latter.

Another important point to note here is,

The product of E and I is called Flexural Rigidity.

This parameter will greatly determine the deflection. As it is inversely proportional to deflection, so a greater value of flexural rigidity will lessen the deflection adversely. Nevertheless, you can achieve this by either using a material of better quality (so that E is higher) or by increasing the depth of the beam as compared to its width (so that I is higher).

Example-4



Find the magnitude of the force 'P' such that the deflection at free end (B) is zero.

So here we have two kinds of load acting on the cantilever beam AB. One is a UDL which will cause a downward deflection (let it be δ_{B1}) in the beam and the other one is a pulling force which will cause an upward deflection (let it be δ_{B2}) in the beam as shown in the Figure 10.

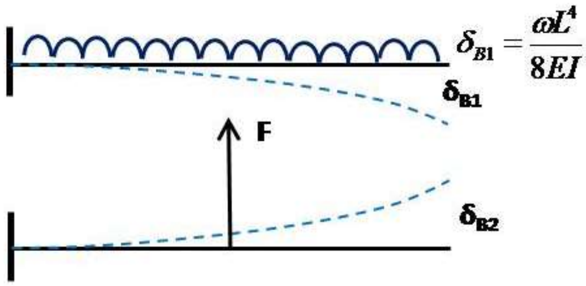


Figure 10 Deflections in case of different kinds of forces

Step-1: Computing deflection in case of the UDL

Given,

w = 30 KN/m

L = 4 m

We know that,

$$\delta_{B1} = \frac{wL^4}{8EI} = \frac{30 \times 4^4}{8EI} = \frac{960}{EI}$$

Step-2: Computing deflection in case of the point load

We know that,

$$\delta_{B2} = \frac{Fa^3}{3EI} + \frac{Fa^2}{2EI}(L-a) \text{ [here a = L/2]}$$

$$= \frac{F}{3EI}\left(\frac{L}{2}\right)^3 + \frac{F}{2EI}\left(\frac{L}{2}\right)^2\left(L-\frac{L}{2}\right)$$

$$= \frac{FL^3}{24EI} + \frac{FL^3}{16EI} = \frac{FL^3}{8EI}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{FL^3}{8EI}\left(\frac{2+3}{6}\right) = \frac{5FL^3}{48EI} = \frac{5F4^3}{48EI} = \frac{20F}{3EI}$$

Step-3: Equating both the deflections

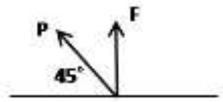
If the deflection at the free end has to be 0 then,

$$\delta_{B1} = \delta_{B2}$$

i.e., $\frac{960}{EI} = \frac{20F}{3EI}$

i.e., F = 144

Again we have,



P cos 45° = F

i.e., P = $\frac{F}{\cos 45°}$ = F√2 = 144 x 1.41 = 203.65 KN

Therefore, I have to put 230 KN of force inclined at an angle of 45° such that the deflection at the free end B will be 0.

Case-4: Simply Supported Beam with Concentrated Load

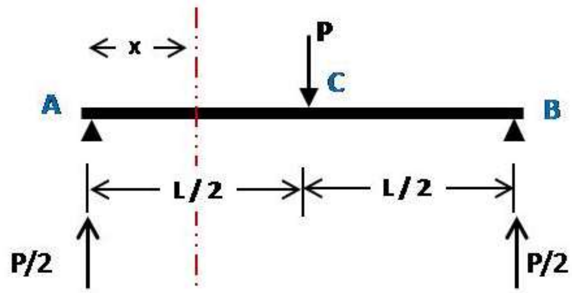


Figure 11 Simply supported beam with concentrated load

Next let us see how to compute the deflection in case of a simply supported beam subjected to a point load. So here we have a simply supported beam AB of length L. A point load P is acting over the beam at its center C.

We know that,

$R_A = R_B = P/2$

Now,

Let us take a section x-x' at a distance of x units from A.

Then, moment at x = $-\frac{P}{2}x$

So,

The moment-curvature equation for the given beam will be

$$EI\frac{d^2y}{dx^2} = -\frac{P}{2}x$$

Integrating the above equation we get,

$EI\frac{dy}{dx} = -\frac{P}{4}x^2 + C_1$ [Slope equation]

Integrating the slope equation we get,

$EIy = -\frac{P}{12}x^3 + C_1x + C_2$ [Deflection equation]

Boundary Condition-1:

At mid-span C we have,

x = L/2, Slope = 0

Substituting these values in the slope equation we get,

$0 = -\frac{P}{4} \times \frac{L^2}{4} + C_1$

i.e., $C_1 = \frac{PL^2}{16}$

Boundary Condition-2:

At A we have,

x = 0, Deflection = 0

Substituting these values in the deflection equation we get,

$0 = \frac{P}{12}0^3 + C_1 \times 0 + C_2$

i.e., $C_2 = 0$

Thus,

We can re-write the equations for slope and deflection as below

$$EI\frac{dy}{dx} = -\frac{P}{4}x^2 + \frac{PL^2}{16}$$

And $EIy = -\frac{P}{12}x^3 + \frac{PL^2}{16}x$, respectively.

But we have,

Slope at A and B (x = 0, L) are equal.

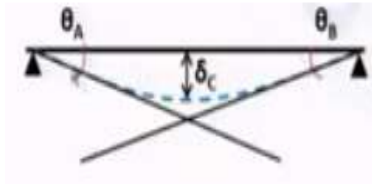
i.e., $\theta_A = \theta_B = \frac{PL^2}{16EI}$

Also,

Deflection at mid-span, C (x = L/2)

So, $EI\delta_C = -\frac{P}{12} \times (\frac{L}{2})^3 + \frac{PL^2}{16} \times \left(\frac{L}{2}\right) = \frac{PL^3}{48}$

i.e., $\delta_c = \frac{PL^3}{48EI}$



Case-5: Simply Supported Subjected to UDL

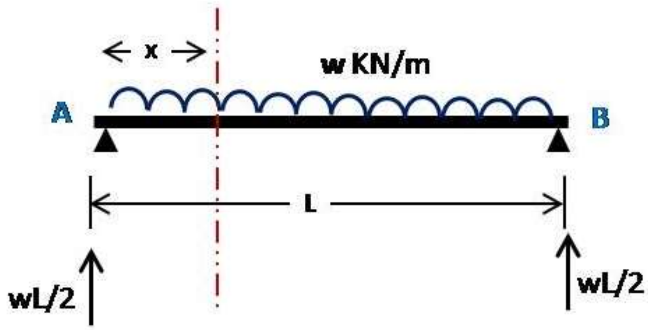


Figure 12 Simply supported beam with UDL

Similarly, here we have a UDL whose Moment-Curvature equation can be given as below:

$$EI\frac{d^2y}{dx^2} = M = -\frac{wL}{2}x + \frac{wx^2}{2}$$

Integrating the above equation we get,

$$EI\frac{dy}{dx} = -\frac{wL}{4}x^2 + \frac{w}{6}x^3 + C_1 \text{ [Slope equation]}$$

Integrating the above equation we get,

$$EIy = -\frac{wL}{12}x^3 + \frac{w}{24}x^4 + C_1x + C_2 \text{ [Deflection equation]}$$

Boundary Condition-1:

We have,

At mid-span, x = L/2 and slope = 0

Substituting these values in the slope equation we get,

$$0 = -\frac{wL}{4} \times (\frac{L}{2})^2 + \frac{w}{6} \times (\frac{L}{2})^3 + C_1$$

i.e., $C_1 = \frac{wL^3}{24}$

Boundary Condition-2:

At A, x = 0 and deflection = 0

Substituting these values in the equation of deflection we get,

$$C_2 = 0$$

Thus, we can re-write the equation of slope as

$$EI\frac{dy}{dx} = -\frac{wL}{4}x^2 + \frac{w}{6}x^3 + \frac{wL^3}{24}$$

But we know that,

Slope at A (x = 0) = Slope at B (x = L)

i.e., $\theta_A = \theta_B$

Substituting x = 0 in the slope equation we get,

$$EI\theta_A = \frac{wL}{4}0^2 + \frac{w}{6}0^3 + \frac{wL^3}{24}$$

i.e., $\theta_A = \frac{wL^3}{24EI} = \theta_B$

Also, we can re-write the equation of deflection as

$$EIy = -\frac{wL}{12}x^3 + \frac{w}{24}x^4 + \frac{wL^3}{24}x$$

Then, deflection at the mid-span (x = L/2)

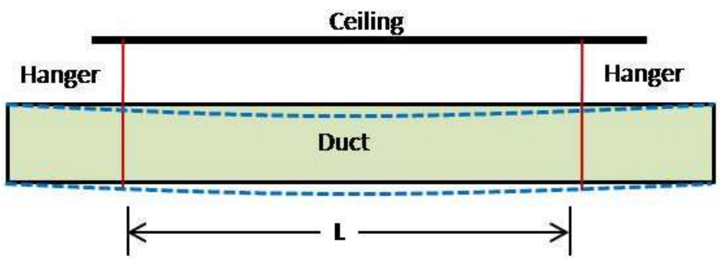
$$EI\delta_{mid} = -\frac{wL}{12} \times (\frac{L}{2})^3 + \frac{w}{24} \times (\frac{L}{2})^4 + \frac{wL^3}{24} \times \frac{L}{2}$$

i.e., $EI\delta_{mid} = -\frac{wL^4}{96} + \frac{wL^4}{384} + \frac{wL^4}{48} = \frac{5wL^4}{384}$

i.e., $\delta_{mid} = \frac{5wL^4}{384EI}$

Example-5

Q. A 600 X 600 mm square air-conditioning duct is made out of 5mm thick aluminum sheet. Find the maximum support span to control the central deflection within (Span/240). Take Density of aluminum as 2700Kg/m^3 and Young's Modulus of Elasticity of aluminum is 1 X 10^4 N/mm^2. Assume the deflection due to self weight of duct only, neglect the weight of air.



Duct is freely supported between two hangers

Here we have,

Cross section area = $(600^2 – 590^2)$ = 11900 mm^2

Moment of Inertia = $(600^4 – 590^4)/12$ = 702.2 X 10^6 mm^4

UDL due to self weight = 11.9 X 10^{-3} X 1 X 2700 X10 = 321.3 N/m = 0.3213 N/mm

Also,

$$\frac{5wL^4}{384EI} = \frac{L}{240}$$

i.e., $L^3 = \frac{384EI}{5\times240w} = \frac{384\times1\times10^4\times702.2\times10^6}{5\times240\times0.3213} = 6.99\times10^{12}$

i.e., L = 19123 mm = 19.1 m

Conclusion

To sum up, I'd like to state the following:

- The slope and deflection of beam due to external loading can be deduced from bending equation.
- Other than the intensity of loading and span of beam, flexural rigidity plays a vital role in beam deflection.

References

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- **Strength of Materials** By B.C. Punmia, Ashok K.Jain & Arun K.Jain Laxmi Publication
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Homework

Q1. A certain intensity of UDL is distributed over a simply supported beam of span 'L'. If the total load (equivalent to the UDL) is now acted as a concentrated load at mid span what will be the change in maximum deflection?

Q2. Find the deflection at free end 'B' of the cantilever beam given in the figure below.