Structural System in Architecture
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Lecture - 17
Indeterminate Beams

Welcome to NPTEL's online certification course on Structural Systems in Architecture. Today we are in the 2nd lecture of module 4 (Frame Structure Analysis and Design) i.e., the Indeterminate Beams.

Concepts Covered

The concepts covered under this lecture are as follows:

- Methods of Structural Analysis
- Degree of Indeterminacy
- Force Method or Flexibility Method
- Displacement Method or Stiffness Method
- Analysis of Continuous Beam
- Analysis of Propped Cantilever

Learning Objectives

The learning objectives of this lecture are given below:

- Discussing the types of Methods of Structural Analysis.
- Differentiating between Force and Displacement Methods.
- Illustrating the Continuous beam and Propped Cantilever.

Introduction

Structural Analysis is an important step towards the overall structural engineering. In Structural Analysis a designer determines the internal forces like axial compression, bending moment, shear force, twisting moment etc. Structural analysis also determines the displacements due to time-independent loading conditions. This process requires the knowledge of mechanics and strength of materials. The output of the analysis is beneficial for the designing process.

The elastic and linear methods of structural analysis are based on following conditions:

- Material of a structure obeys Hooke's law.
- Displacements of a structure are small.
- Parameters of a structure do not change under loading.

Methods of Structural Analysis

There are various types of methods of structural analysis broadly classified under four categories. These methods under their respective classifications are mentioned below:

1. Classical methods
    a. Three-moment method
    b. Consistent deformation method
    c. Slope-deflection method
    d. Methods of strain energy

2. Computer methods
    a. Matrix method
    b. Finite difference method
    c. Finite element method

3. Relaxation/Iterative methods
    a. Moment distribution
    b. Method
    c. Kani's method

4. Approximate methods
    a. Substitute frame method
    b. Portal method
    c. Cantilever method

It is important to note that the iterative method, as the name suggests, involves lot of iterations. Hence complicated structures cannot be solved with these methods. Conversely, the approximate methods give very quick results. However, the results will be approximate of course.

Nonetheless, in this course we'll not be dealing with all of these methods. Our concern here is to study how an indeterminate beam can be solved from the architectural point of view and after solving what is the tendency of the bending moment and how can it be conceived in the design.

Before moving to the next topic it is important to mention here that Statically Indeterminate Structure can be analyzed by various methods. But by virtue of the theory and approaches the methods are classified into following two major types:

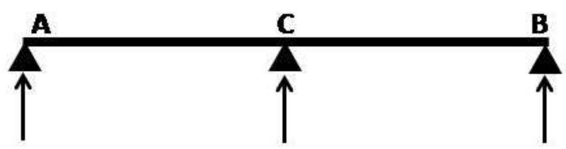
1. Force Method of Analysis
2. Displacement Method of Analysis

Degree of Indeterminacy

We've already mentioned in our lectures of the first week that Degree of Indeterminacy is the number of unknown equations in a structure over the available equilibrium equations. By the term'unknown equations' we mean the reaction unknown equations. As the number of supports increases so do the reaction forces from these supports. Those are unknowns and hence we need to find them.

Besides, we also have to look for the known equations. In general, we might have around three available equilibrium equations viz., $\sum Fx = 0$, $\sum Fy = 0$ and $\sum M = 0$. Apart from these some more, maybe one or two, equations can be there also depending on the specific cases. Sometimes there might be any internal hinge or link also though it's less less likely to occur in reinforced concretes or steel structures or any other structure to be studied for the purpose of Civil Engineering or Architecture.

However, if the number of unknowns become more than the knowns then the structure becomes statically indeterminate. In other words, it cannot be achieved by the statical equations of equilibrium. So the difference between the unknowns and the available equations is the degree of indeterminacy. Now let us see some examples below where the unknowns have been denoted as U, the available equilibrium equations have been denoted as E and the degree of indeterminacy have been denoted as DOI.



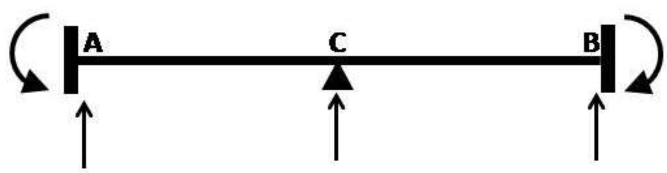
U = 1+1+1 = 3

E = 2

DOI = (3-2) = 1

Figure 1 Continuous beam

Given in the Figure 1 is a continuous beam. Assuming all the forces are gravity load, no forces have been considered in the x-direction. Hence, $\sum Fx = 0$ will not come into picture. Thus the only known equations are $\sum Fy = 0$ and $\sum M = 0$.



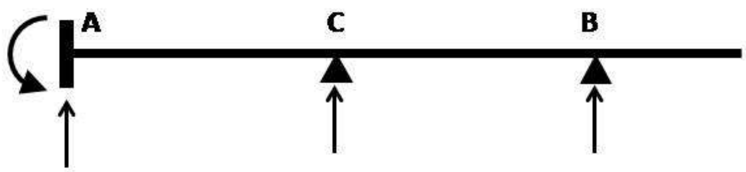
U = 2+1+2 = 5

E = 2

DOI = (5-2) = 3

Figure 2 Fixed beam

Given in the Figure 2 is a fixed beam. In case of fixed beams there are always two additional equations as there are moments at both the ends which are unknowns.



U = 2+1+1 = 4

E = 2

DOI = (4-2) = 2

Figure 3 Fixed, continuous and overhang beam

The Figure 3 shows a beam which is fixed from one end and then it is continuous up to some extent and then ending in an overhang.

To put it differently, in the first case you've to find out one known equation to solve the structure because whenever you'll proceed to calculate the bending moment and all, the first thing you'd need are the reactions. Without knowing the reactions you cannot proceed further.

In the second case you need three such equations to solve the structure. These equations can be formulated using the theories of Structural Analysis. In the third case you need two such equations. These equations can be found out using some of the methods which we had briefly mentioned earlier and will discuss in detail now.

Force Method or Flexibility Method

In the force method of analysis, the primary unknowns are the forces acting upon the structure in concern. For instance, if you exert a force of P units on an elastic body, it is going to deform. Besides, there'd be some stiffness K in the body. So, the deflection will be P/K. Thus, here we can say that in order to find the deflection, you've to first know the K if P is already known or vice versa. Similarly, in this method the primary unknowns are the forces.

In this method compatibility equations are written for displacement and rotations (which are calculated by force displacement equations). Redundant forces are calculated while solving these equations. Once the redundant forces are calculated, the remaining reactions are evaluated by equations of equilibrium. Some of the examples of the force method are given below:

- Method of Consistent Deformation
- Theorem of least work
- Column Analogy method
- Three moment Equation Method
- Unit Load Method
- Flexibility Matrix Method

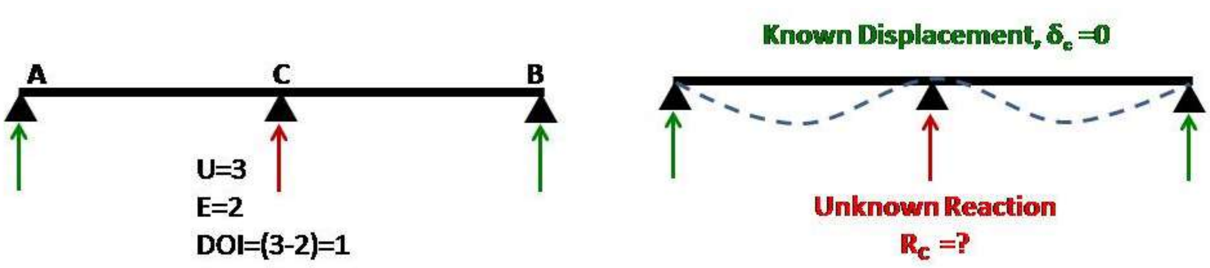


Figure 4 Given continuous beam and its deflection

In the Figure 4 we have a continuous beam ACB in which the reaction at C is unknown. But if you consider its deflected state you'll realize that in its deflected state the displacement at C is known and that is 0.

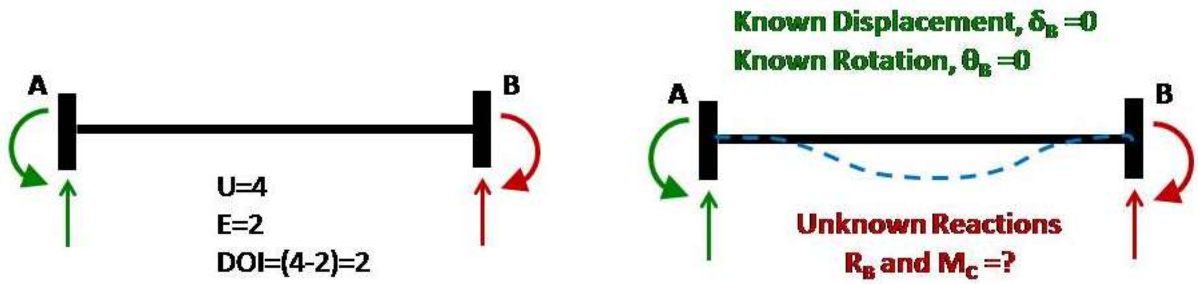


Figure 5 Given fixed beam and its deflection

Similarly, in the fixed beam AB given in Figure 5 the reaction and moment at B are unknown. But when you consider its deflected state you'll get two known equations concerning the point B as the displacement and slope at this point are 0.

So these extra equations that we got in both the cases are known as the compatibility equations. It is beneficial in finding the unknowns present in any structure. We'll deal with more such examples in this lecture.

Displacement Method or Stiffness Method

In the displacement method of analysis, the primary unknowns are the displacements. In this method, force -displacement relations are computed at first and subsequently equations are written satisfying the equilibrium conditions of the structure. After determining the unknown displacements, the other forces are calculated complying with the compatibility conditions and force displacement relations. The displacement-based method is amenable to computer programming and hence the method is being widely used in the modern-day structural analysis. Following are some of the examples of the displacement method.

- Slope-Deflection method
- Moment Distribution method
- Kani's method
- Stiffness Matrix method

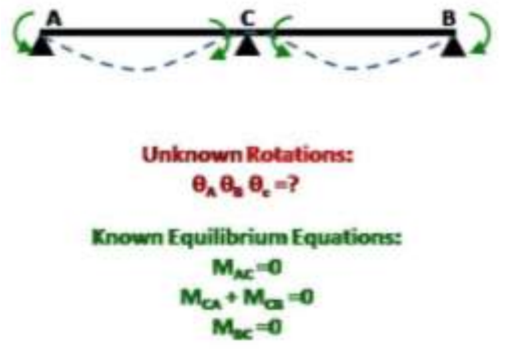


Figure 6 Equilibrium equations in a continuous beam

Here the rotations at A, B and C are unknown but the moments at these points are known. So these become the equilibrium equations which would be useful in solving the unknowns.

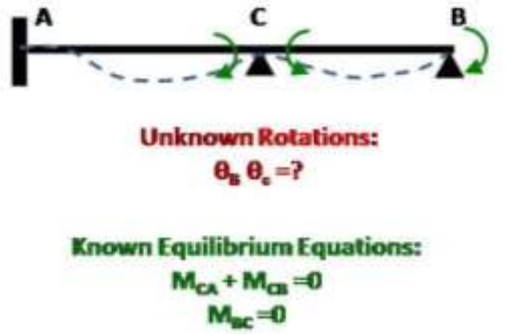


Figure 7 Equilibrium equations when one end of the beam is fixed

Similarly, when one end is fixed we cannot say that the moment at that end is 0, so here we have only two equilibrium equations which are enough to find the unknowns as the unknowns are also two in number, i.e., the rotations at B and C. At A there won't be any rotation as it is fixed.

Analysis of Continuous Beam

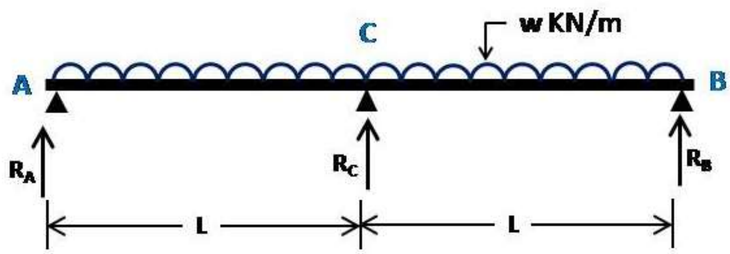


Figure 8 Continuous beam loaded with UDL

Here we have a beam shown in the Figure 8 for which the following conditions hold true:

Unknown Reaction = 3

Static Equilibrium equation available = 2

Degree of Indeterminacy of the beam = 1

Therefore, only *one* Compatibility Equation is required to solve the third unknown reaction.

Here the given continuous beam can be broken into two separate determinate beams and superimposed to find the value of one unknown reaction (lets R_c) as shown below in the Figure 9.

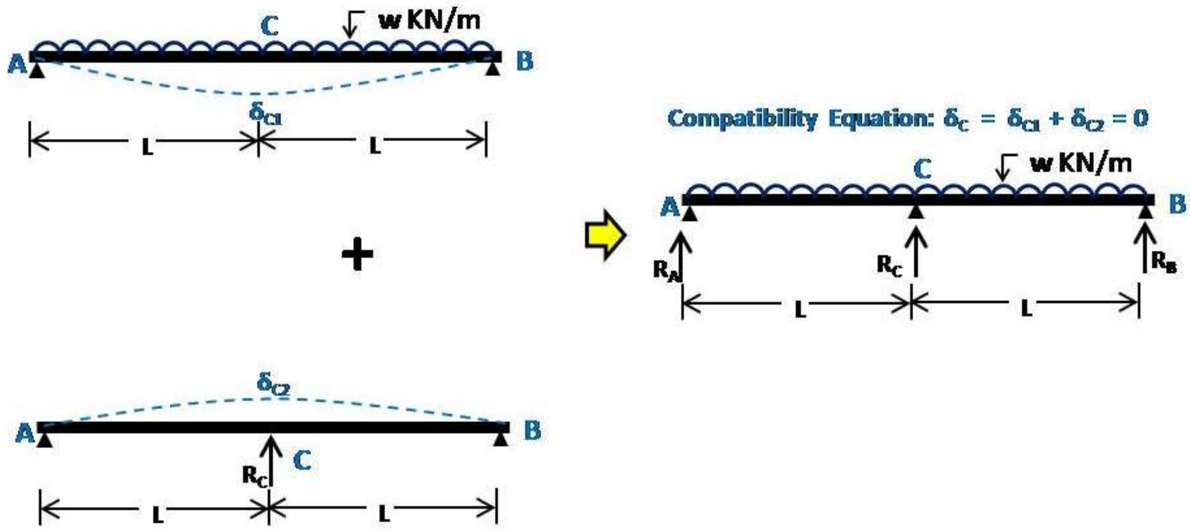


Figure 9 Solving the beam given in Figure 8

As evident from the Figure 9, the first portion has now become a simply supported beam AB subjected to UDL of w KN/m. Let the deflection here be termed as δ_{C1}. Secondly, the other portion is also a simply supported beam AB but with a point load acting upwards at C. Let this deflection be termed as δ_{C2}. The combination of these two beams will give the original beam and the sum of their respective deflections will give the resultant deflection of the original beam, i.e. 0.

Hence, the compatibility equation here is

$$\delta_C = \delta_{C1} + \delta_{C2} = 0$$

We have downward deflection at 'C' due to UDL:

$$\delta_{C1} = \frac{5}{384} \times \frac{w(2L)^4}{EI} = \frac{5wL^4}{24EI} \ (\downarrow) \text{ [From Lecture 16]}$$

And upward deflection at 'C' due to Unknown Reaction R_c :

$$\delta_{C2} = \frac{1}{48} \times \frac{R_C(2L)^3}{EI} = \frac{R_C L^3}{6EI} \ (\uparrow)$$

But we know that,

$$\delta_{C1} = \delta_{C2}$$

i.e., $\frac{5wL^4}{24EI} = \frac{R_C L^3}{6EI}$

i.e., $R_c = \frac{5wL}{4}$

We also know that,

$$R_A + R_C + R_B = 2wL$$

i.e., $R_A + \frac{5wL}{4} + R_A = 2wL$ [Since $R_A = R_B$]

i.e., $2R_A = 2wL - \frac{5wL}{4}$

i.e., $R_A = \frac{3wL}{8}$

Now let us find the point where shear force is 0. So, let the point be situated at x units from A.

Then, $\frac{3wL}{8} - wx = 0$

i.e., $x = \frac{3L}{8} = 0.375L$

Thus, the resultant SFD is given in the Figure

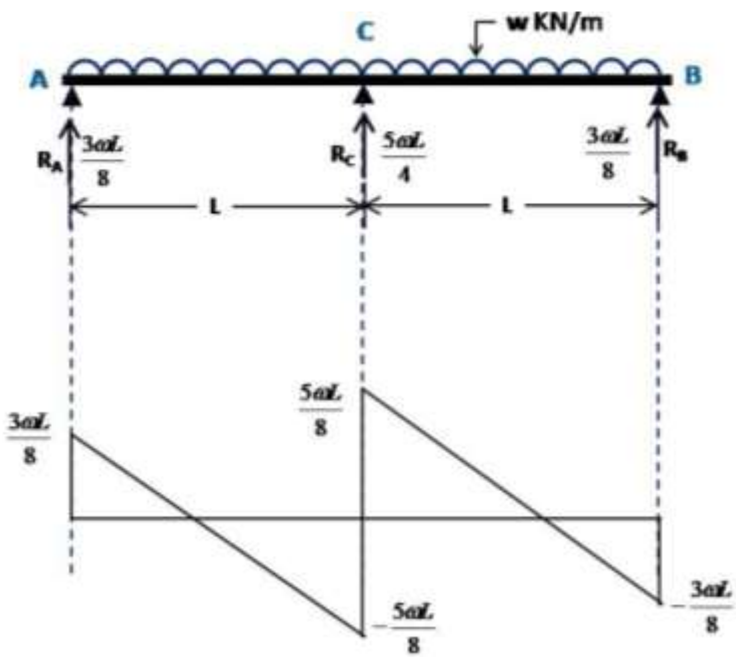


Figure 10 SFD of the given beam

Next,

We have both the supports at the ends of the beam hinged.

i.e., $M_A = M_B = 0$

Now,

Moment at any section at a distance ‘x’ from A, $M_x = \frac{3wL}{8}x - \frac{w}{2}x^2$

And we have,

$M_c = \frac{3wL}{8}L - \frac{w}{2}L^2 = -\frac{wL^2}{8}$

Next we have,

Shear force is zero at $x = \frac{3L}{8} = 0.375L$

But we know that,

Maximum bending moment will occur at the point where shear force is zero.

So, $M_{max} = \frac{9wL^2}{128}$ [By substituting the value of x in the expression of M_{max}]

Now for finding the point of contra-flexure,

$M_x = 0$

i.e., $x = \frac{3}{4}L = 0.75L$ [By equating the expression of M_x with 0]

Therefore, the resulting BMD is given in the Figures 11 and 12.

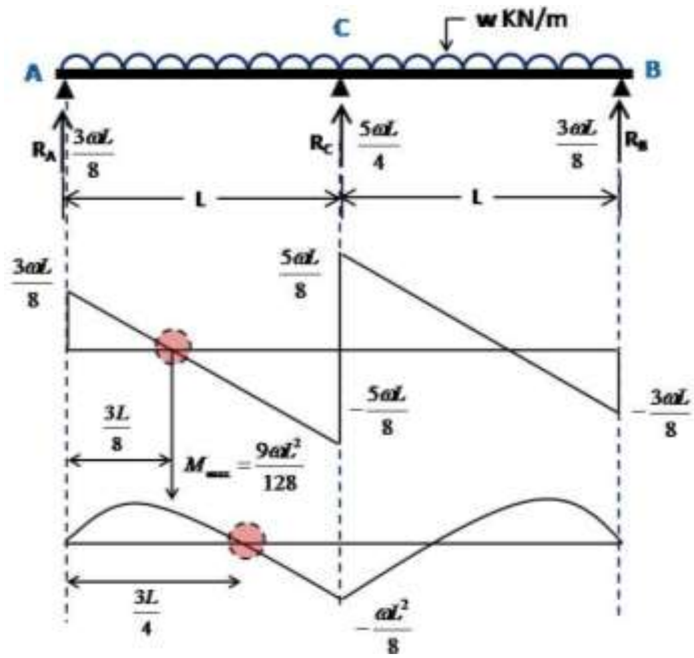


Figure 11 BMD of the given beam

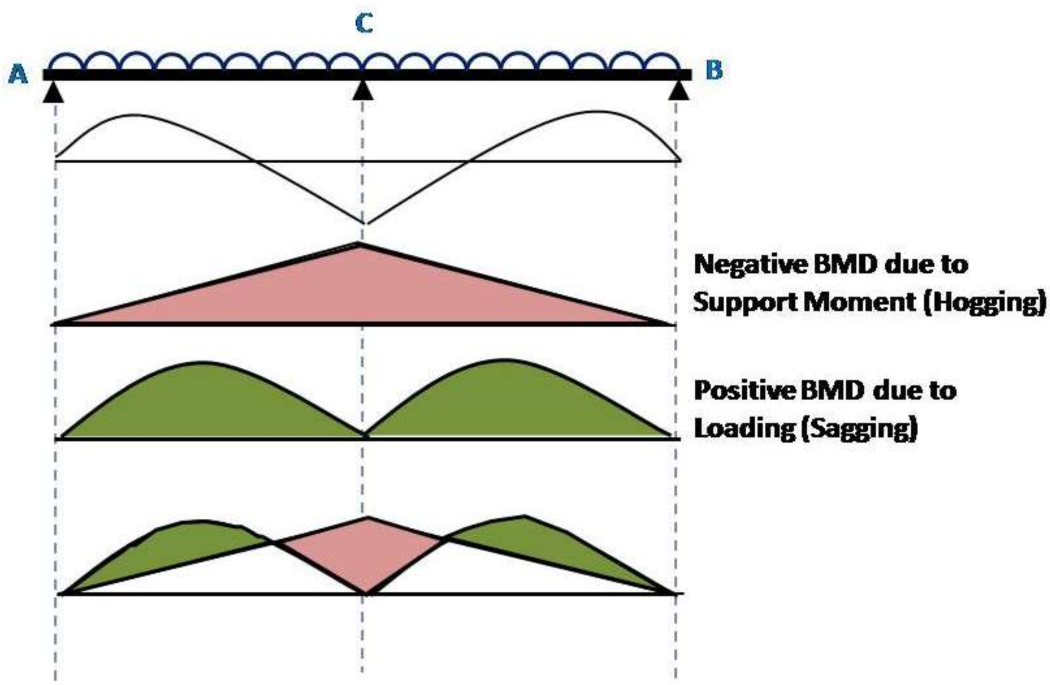


Figure 12 Understanding the sagging and hogging moments in the BMD

Analysis of Propped Cantilever Beam

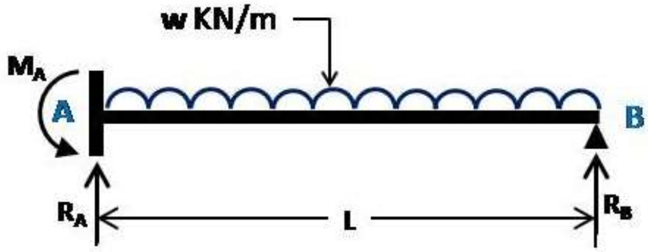


Figure 13 Propped cantilever beam

Given in the Figure 13 is a propped cantilever beam of two equal span (L) is loaded with UDL of intensity 'w' where the following conditions hold true:

Unknown Reaction = 3

Static Equilibrium equation available = 2

Degree of Indeterminacy of the beam = 1

Therefore, ONE Compatibility Equation is required to solve the third unknown reaction, and let it be Reaction at B (R_B)

Here the given beam can be disintegrated as shown in the Figure 14.

Figure 14 Disintegration of the given beam

$\delta_{B1} = \frac{wL^4}{8EI} (\downarrow)$ $\delta_{B2} = \frac{R_B L^3}{3EI} (\uparrow)$

We know that,

$$\delta_B = \delta_{B1} + \delta_{B2} = 0$$

i.e., $\frac{R_B L^2}{3EI} = \frac{wL^4}{8EI}$

i.e., $R_B = \frac{3wL}{8}$

Also, $R_A + R_B = wL; i.e., R_A = wL - \frac{3wL}{8} = \frac{5wL}{8}$

Then, shear force is 0 at:

$$wx = \frac{3wL}{8}$$

i.e., $x = \frac{3L}{8}$

Then the final shear force diagram can be drawn as given in Figure 15.

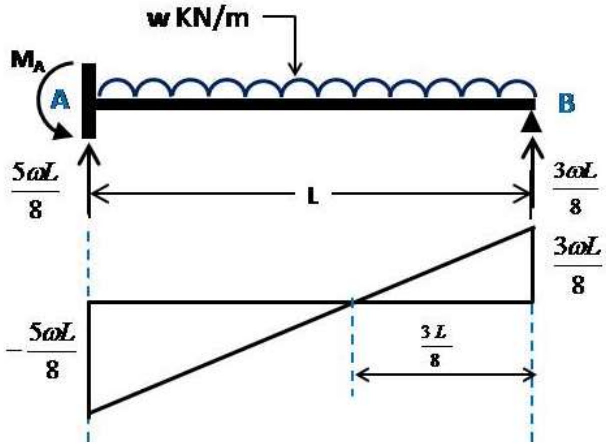


Figure 15 SFD of the given beam.

As shear force is 0 at x = $\frac{3L}{8}$ from the free end, so bending moment has to be maximum at this point. Thus,

$$M_{max} = \frac{3wL}{8}x - \frac{w}{2}x^2 = \frac{3wL}{8} \times \frac{3L}{8} - \frac{w}{2} \times (\frac{3L}{8})^2 = \frac{9wL^2}{128}$$

Then, $M_A = \frac{3wL}{8}L - \frac{w}{2}L^2 = -\frac{wL^2}{8}$

Point of contra-flexure:

$$M_x = 0$$

i.e., $\frac{3wL}{8}x - \frac{w}{2}x^2 = 0$

i.e., $\frac{3wL}{8}x = \frac{w}{2}x^2$

i.e., $x = \frac{3L}{4}$

Therefore, the final BMD is given in the Figure 16.

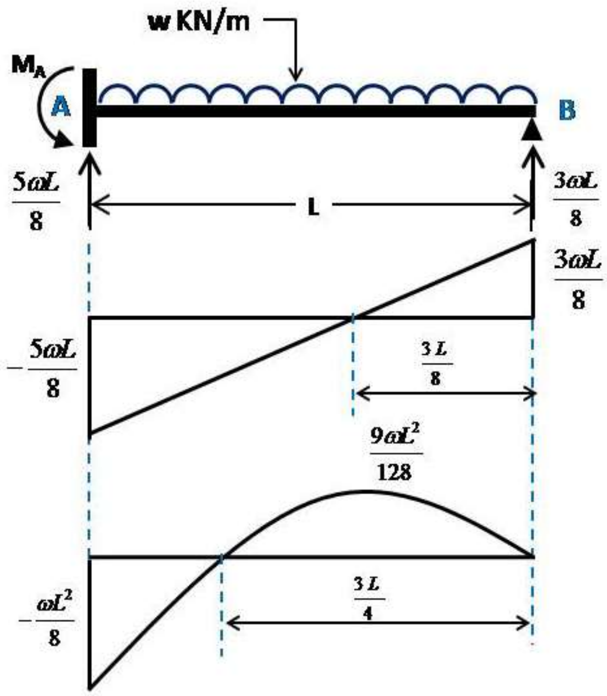


Figure 16 BMD of the given beam

References

- **Engineering Mechanics** by Timishenko and Young McGraw-Hill Publication
- **Strength of Materials** By B.C. Punmia, Ashok K.Jain & Arun K.Jain Laxmi Publication
- **Basic Structures for Engineers and Architects** By Philip Garrison, Blackwell Publisher
- **Understanding Structures:** ***An Introduction to Structural Analysis*** By Meta A. Sozen & T. Ichinose, CRC Press

Conclusions

To conclude I'd like to state the following:

- The indeterminate structures are having excess reaction unknowns over the available static equilibrium equations.
- Additional compatibility or equilibrium equation is developed to solve the structure.
- Methods of structural analysis are broadly classified into Force and Displacement Methods.

Homework

Q1. Sketch the Shear Force and Bending Moment Diagram profile of the following beams.