Structural System in Architecture
Prof. Shankha Pratim Bhattacharya
Department of Architecture and Regional Planning
Indian Institute of Technology, Kharagpur

Module No # 02
Lecture No # 07
Theory of Elasticity-2

Welcome to the NPTEL online certification course on structural systems in architecture. Today we will discuss the lecture number 7 this is in the module 2 in the Strength of Materials. The topic of this lecture is Theory of Elasticity-2. If you remember in the part 1 we discussed the Hooke's law and the elasticity theory and with that we also discussed the stress-strain curve of an elastic material.

Concepts Covered

In this lecture we will delve deeper into the topic with the following concepts:

- Poisson's Ratio
- Shear Stress and Rigidity Modulus
- Bulk Modulus
- Engineering Properties of Material
- Factor of Safety
- Concept of Modular Ratio

Learning Objectives

The learning objectives of the lecture are as follows:

- To define the elastic constants.
- To understand the application of factor of safety.
- To learn the concept of modular ratio and its association in design.

Poisson's Ratio

In mechanics, Poisson's ratio is the negative of the ratio of transverse strain to lateral or axial strain. It is named after Siméon Poisson and denoted by the Greek letter 'μ'. It is the ratio of the amount of transversal expansion to the amount of axial compression for small values of these changes.

Hence, it is defined as follows:

"The ratio of Lateral Strain to Longitudinal Strain is always constant for a given material and is known as the Poisson's Ratio."

Mathematically,

Poisson's Ratio (μ) = $\frac{Lateral\ Strain}{Longitudinal\ Strain}$

Poisson's Ratio is dimensionless and for most materials, it ranges between 0-0.5.

When a material is stretched in one direction, it tends to compress in the direction perpendicular to that of force application and vice versa. The measure of this phenomenon is given in terms of Poisson's ratio. For example, a rubber band tends to become thinner when stretched. For tensile deformation, Poisson's ratio is positive whereas, for compressive deformation, it is negative.

To understand this concept better, let as imagine a material such as rubber as shown below in the Figure 1(a). Now let us suppose that this material is pulled from its ends (lateral strain occurs) as shown in the Figure 1(b).

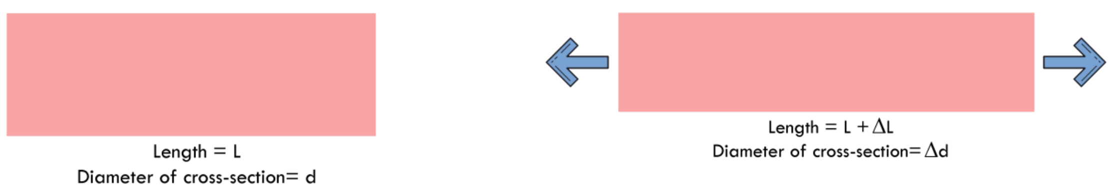


Figure 1 (a)- Original Bar; (b)- Transformed Bar

As shown in the figures, when the tensile force is applied at the ends of the bar of length L, it elongates by a length of ΔL whereas, the force results in a compression in the vertical direction of the bar (longitudinal strain occurs) which in return decreases the original diameter (d) of the cross-section by Δd.

Formula of Poisson's Ratio

Given in the Figures 2(a) and 2(b) are two pictures of the same bar suspended vertically, before and after transformation due to the application of load respectively.

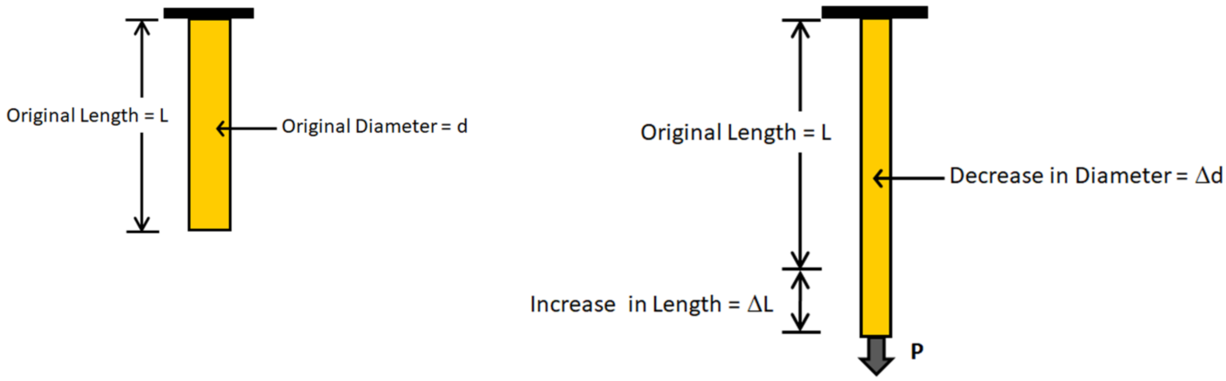


Figure 2 (a)- Original Bar; (b)- Transformed Bar

Contrary to the previous case, here, the strain acting upon the cross-section of the bar is lateral strain whereas, that acting upon the length of the bar is longitudinal strain.

Hence, Poisson's Ratio (μ) = $\frac{Lateral\ Strain}{Longitudinal\ Strain}$

But, lateral strain = $\frac{change\ in\ diameter}{original\ diameter}$

And, longitudinal strain = $\frac{change\ in\ length}{original\ length}$

Here we have,

Original diameter = d

Change in diameter = Δd

Original length = L

Change in length = ΔL

Then, lateral strain = $\frac{\Delta d}{d}$

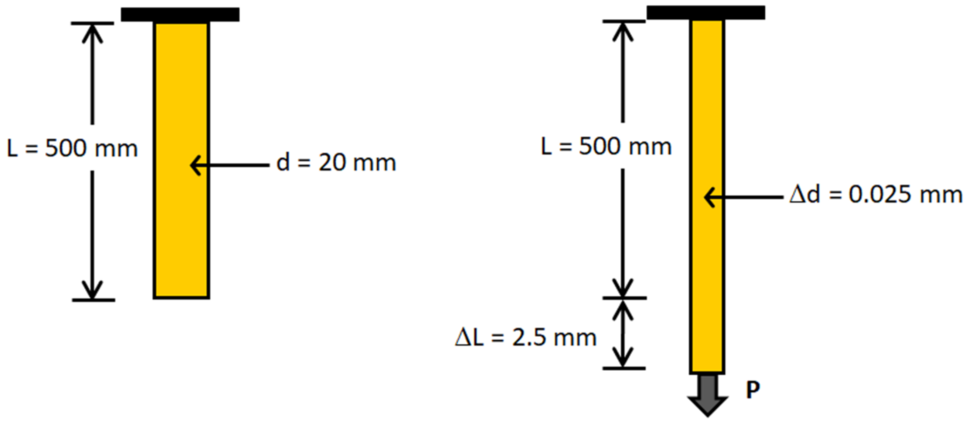
And, longitudinal strain = $\frac{\Delta L}{L}$

Therefore,

$$\mu = \frac{\frac{\Delta d}{d}}{\frac{\Delta L}{L}} = \frac{\Delta d}{d} \times \frac{L}{\Delta L}$$

Numerical-1

Q. A rod of original length 500mm is elongated by 2.5mm under a tensile load. During the process the diameter of the rod is reduced by 0.025mm. The original diameter of the rod was 20mm. Compute the Poisson's Ratio of the material.



Solution:

Given,

L = 500 mm

ΔL = 2.5 mm

Δd = 0.025 mm

d = 20 mm

Then,

Step-1: Find the lateral strain.

Lateral strain = $\frac{\Delta d}{d} = \frac{0.025}{20}$ = 0.00125

Step-2: Find the longitudinal strain.

Longitudinal strain = $\frac{\Delta L}{L} = \frac{2.5}{500}$ = 0.005

Step-3: Calculate the Poisson's Ratio.

Poisson's Ratio (μ) = $\frac{0.00125}{0.005}$ = 0.25

Therefore the value of the Poisson's ratio for the given criteria is 0.25.

Shear Stress

You may have tried to break a thick wooden stick, but failed in the attempt to do so. You may have tried to break it by stepping on it really hard. Here, what made it break is its shear stress.

Shear stress is the deforming force acting per unit area and in the direction perpendicular to the axle of the member. The impact of your load when you step in a wooden stick causes two types of stress, these are:

- Bending Stress, which is parallel to the axle of the member also called flexural stress.
- Shear Stress, which acts in a direction perpendicular to the axle of the member.

Nonetheless, it is interesting to note that the word "shear" means 'to cut off'. When force is applied over the surface area of a rigid body (force acting in a direction parallel to the surface) then this force tries to cut off one part of the body from the other. As a result of this the body gets deformed and hence strain is produced (shear strain- the angular deflection of the body from its original position). Due to the rigidity of the body, it resists the deformation caused and a restoring force (equal and opposite to the applied force) is developed along the surface of the body as per Newton's 3^{rd} law of motion). This restoring force of the body tends to oppose the shearing effect of the applied force. Thus shear stress is just an effect of shear strain.

Hence, the shear force may be defined as:

"It is a force that acts on a plane which passes through the body."

Mathematically,

Shear stress = q = $\frac{force}{area}$

It is measured in N/mm^2.

The shear forces are unaligned and separate the structure into two different parts in opposite directions. The shear force acts in a perpendicular direction to the larger part of the body.

Shear force, in a beam, acts perpendicular to the longitudinal axis. The beam's ability to resist shear force is much more important as compared to its ability to resist axial force. Axial force acts parallel to the longitudinal axis of the beam.

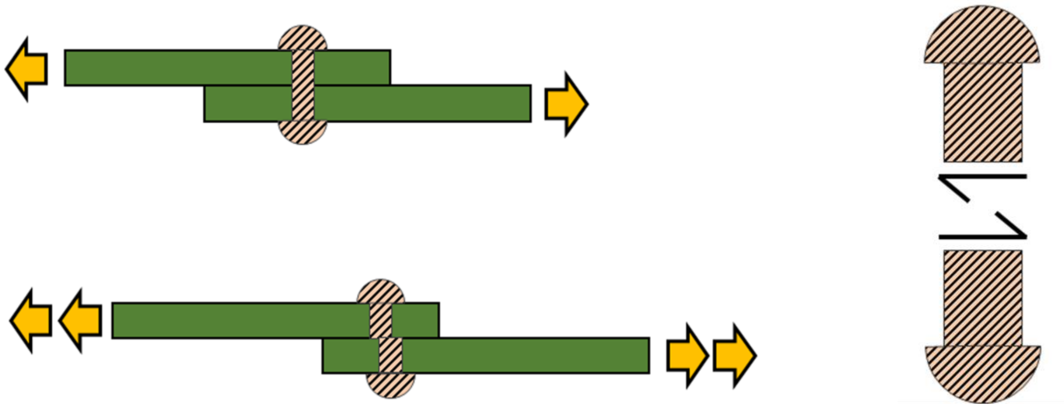


Figure 3 (From top right) (a)- Two plates fastened by a rivet; (b)- Shear stress due to breakage of rivet due to pulling in the opposite directions; (c)- a broken rivet

To better explain the shear force let us consider two identical plates as shown in the figure above. Figure 3(a) shows two green coloured plates fastened together by a rivet. Now let us suppose that the two plates are pulled in two opposite directions as shown in the Figure 3(b). Under this circumstance the excessive shear force caused will result in the slicing of the rivet as shown in the Figure 3(c). Hence this force is also known as a slicing force and acts in a direction perpendicular to the larger part of the body.

However, it is important to note that the rivet could break only because both the plates were of equal dimension and hence the force acting upon was also equal and opposite.

Deformation under Shear Stress

Let us consider a block made of any material or best, let us consider a very thick book to better understand this concept. Then, when you subject it to a tangential force from the top, the book will deform as shown in the figure 4.

This kind of deformation takes place when the force is applied at the top tangentially as shown in the Figure 4 the shear force is transferred through the different layers of the book or the block. The change in angle at the corner of the original block or book is called the shear strain. Shearing forces cause shearing deformation. An element subjected to shear does not change in length but undergoes a change in shape.

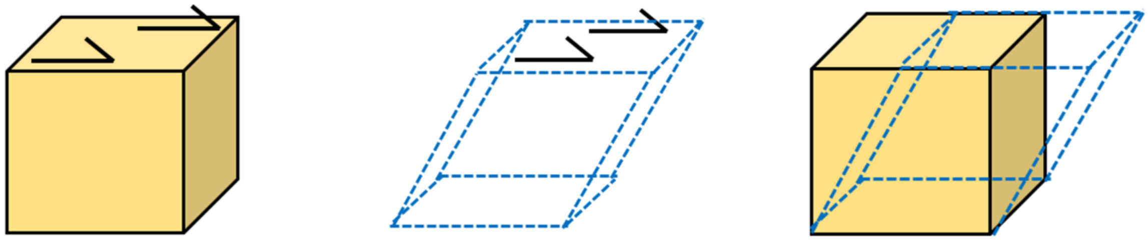


Figure 4 Deformation of a block under shear stress

Modulus of Rigidity

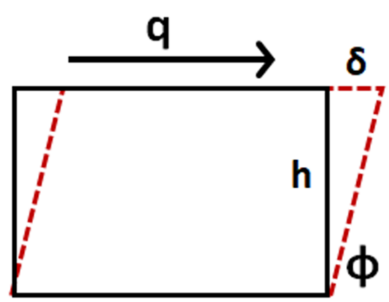


Figure 5 Modulus of Rigidity

Figure 5 shows a one sided sectional view of the block shown in Figure 4 which is having a height 'h' and a shear stress 'q' units is applied at the top. So because of this application of the shear stress at the top there will also be resultant shear stress acting in the base. This eventually results in the deformation of the body as represented by the red dotted lines in the Figure 5.

We can measure the deformation by the angular displacement as Φ. But this angle is so small that tan Φ is equivalent to Φ.

Mathematically,

Since, $\tan \Phi \rightarrow \Phi$

Therefore, $\tan \Phi = \Phi = \frac{\delta}{h}$ = Shear Strain

Where, δ is the small increment of the lateral displacement at the top most layer and h is the thickness of the plate or thickness of that particular block.

Then,

Modulus of Rigidity or Shear Modulus (G) = $\frac{q}{\Phi}$

Numerical-2

Q. A 2mX2m metal plate is subjected to a shear force of 40000KN. The lower face of the metal plate is fixed. The thickness of the plate is 200mm. Due to the application of the shear force the top face is deformed by 0.24mm. Find Modulus of Rigidity of the metal.

Solution:

Given,

Dimension of the plate = 2m x 2m

Thickness of the plate, h = 200 mm

Shear force = 40,000 KN

Linear displacement, $\delta = 0.24$ mm

Now,

Step-1: Find the Shear Stress

Shear stress, q = $\frac{shear\ force}{area} = \frac{40{,}000 \times 10^3}{2000 \times 2000}$ = 10 N/mm^2

Step-2: Find the Shear Strain

Shear strain, $\Phi = \frac{deformation}{thickness} = \frac{\delta}{h} = \frac{0.24}{200}$ = 0.0012

Step-3: Calculate the Modulus of Rigidity

Modulus of Rigidity, G = $\frac{q}{\Phi} = \frac{10}{0.0012} = 8.33 \times 10^3$ N/mm^2

Therefore, the modulus of rigidity of the given metal plate was found to be 8.33×10^3 N/mm^2.

Bulk Modulus

Bulk Modulus is the third kind of modulus after the Young's Modulus of Elasticity which is applicable for linear stress or strain, and the Rigidity Modulus which is applicable for the shear strain. The Bulk Modulus is volumetric where considering an object such as a balloon, if you push or press it with some load or may be some stress, the total volume of the balloon decreases or the stress increases internally.

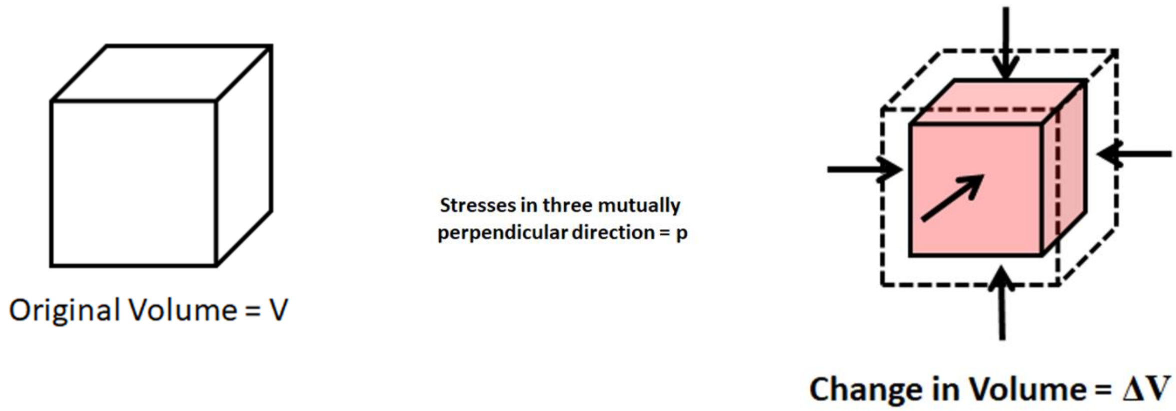


Figure 6 Bulk Modulus

Since the volume of the balloon is decreasing, its volumetric changes can only be computed with the help of some kind of modulus which in this case is the Bulk Modulus.

So let us consider a cube with its original volume V as shown in the Figure 6. And let there be ‘p’ unit of stress acting in three mutually perpendicular directions on the cube.

Let us consider that the cube is filled with air just like a balloon. Then, if the cube is placed under water there would be hydrostatic force acting on the cube from all directions, trying to compress it. At this point, the cube offers some resistivity so as not to get crushed completely. This resistivity is inherent to the material and is known as Bulk Modulus.

Now let us assume that this phenomenon leads to a reduction in the volume of the cube by ΔV. Then,

Volumetric Strain $= \frac{\Delta V}{V}$

Hence, Bulk Modulus, K $= \frac{p}{\frac{\Delta V}{V}} = \frac{pV}{\Delta V}$

Therefore, Bulk Modulus may be defined as

The measure of how resistant the material is to compression.

Mathematically,

$$\text{Bulk Modulus} = \frac{direct\ stress}{volumetric\ strain} = \frac{p}{\frac{\Delta V}{V}} = \frac{pV}{\Delta V}$$

Relationship with other Moduli

Relationship between Young's Modulus and Bulk Modulus:

The relationship between the Young's Modulus of Elasticity and the Bulk Modulus is given as

$$E = 3K\,(1\text{-}2\mu)$$

Where, E is the Young's Modulus, K is Bulk Modulus and μ is the Poisson's Ratio.

Relationship between Young's Modulus and Rigidity Modulus:

The relationship between the Young's Modulus and the Rigidity Modulus is given as

$$E = 2N\,(1 + \mu)$$

Where, E is Young's Modulus, N is the Rigidity Modulus and μ is the Poisson's Ratio.

Engineering Properties of Various Materials

Table 1 shows the various engineering properties of different materials, such as the Young's Modulus of Elasticity, Yield Strength and the Ultimate Strength. However, it is important to note that some alloys such as steel varies in its magnitude yield strength and ultimate strength depending on its type. For instance, the stainless steel may have a different value than its high tensile variants. Hence, depending upon the percentage of the carbon used and how the steel is processed, it will have different values of various engineering properties.

Factors of Safety

Here, the graph shown in the Figure 7 is a stress-strain curve of steel. As discussed earlier, there is a point in the curve called the yield stress which comes at a certain distance after the proportional limit.

At this point if you increase the stress or replace the sample with a higher load the stress will not increase but the strain would increase rapidly.

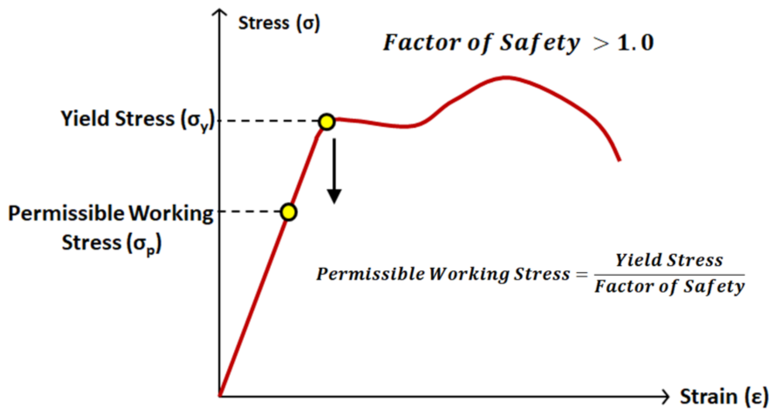


Figure 7 Stress-strain curve of steel showing factor of safety

However, the stress again increases after a while and then it reaches to its ultimate stress and finally the material ruptures. Hence, the yield stress has been considered as one of the important criteria for designing any structure.

Thereupon, during the designing process, it is taken care that the load of the structural element doesn't exceed the yield stress otherwise the material would exhibit plastic deformations which will not be suitable for use in a structural component. Despite this fact, it is advisable that the load restrictions should be fixed after considering a safety margin and not the yield stress straightaway. This stress limit is known as the permissible working stress.

For instance, let us suppose that the yield stress is 250 units but the permissible working stress is 150 units then even though the stress exceeds 150 units and goes up to 160 units or 170 units, the structure will still be safe and would not undergo plastic deformation.

Hence, we may define the Permissible Working Stress as

"The safe stress taken within the elastic range of material."

It is also called the safe stress, working stress, actual stress and allowable stress.

Whereas, Factor of Safety may be defined as

"The number used to determine the working stress."

It is fixed based on the experimental works on the material.

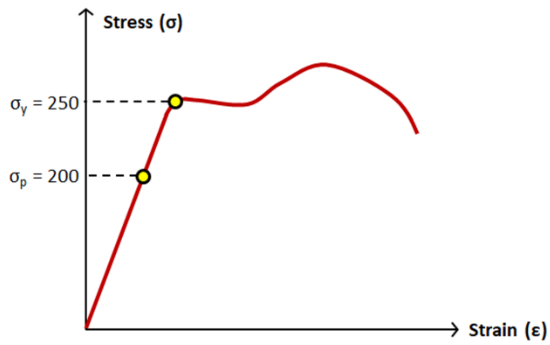
Mathematically,

Permissible Working Stress $(\sigma_p) = \frac{yield\ stress}{factor\ of\ safety}$

Where, factor of safety > 1.

Numerical-3

Q. A steel bar needs to carry an axial pull of 80KN. The Yield Stress of steel is found as 250 N/mm^2. Using Factor of Safety as 1.25, estimate the diameter of the bar required to safely carry the axial pull.



Solution:

Given,

Force, p = 80 KN

Yield stress, σ_y = 250 N/mm^2

Factor of safety = 1.25

Diameter of bar = ?

Then,

Step-1: Find the Permissible Working Stress

Permissible working stress $(\sigma_p) = \frac{yield\ stress}{factor\ of\ safety} = \frac{250}{1.25}$ = 200 N/mm^2

Step-2: Calculate the cross-sectional area of the bar

Required cross-sectional area of the bar $= \frac{p}{\sigma_p} = \frac{80 \times 10^3}{200}$ = 400 mm^2

Step-3: Compute the diameter of the bar

We have,

$$3.142 \text{ x } \frac{d^2}{4} = 400$$

$$\text{i.e., } d^2 = \frac{400 \times 4}{3.142} = 509.09$$

$$\text{i.e., } d = 22.56 \text{ mm}$$

Therefore, the diameter of the bar required to safely carry the axial pull is 22.56 mm.

Percentage Elongation

Consequently, let us now study how to calculate the percentage elongation of a bar when subjected to a tensile stress.

Let us suppose that the original length of a bar is L_0 and at the time of rupture the bar elongates to a length of L_f.

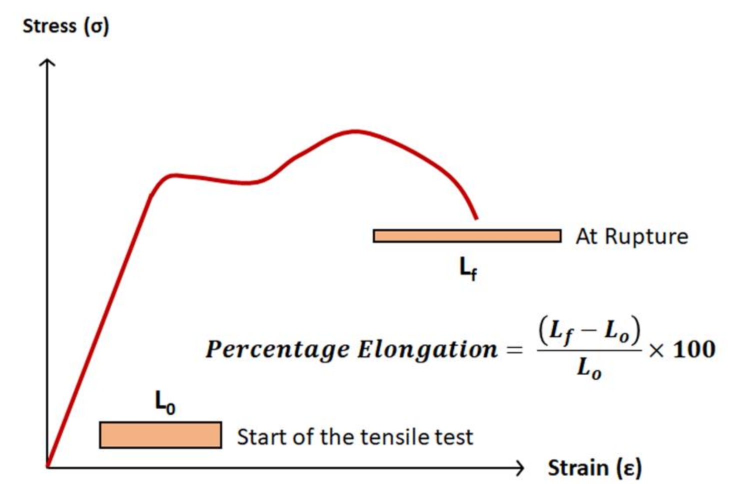


Figure 8 Percentage elongation

Then,

elongation in the bar = $L_f - L_0$

Hence,

$$\text{Percentage elongation } = \frac{L_f - L_0}{L_0} \times 100$$

Types of Fracture

Next, there are mainly three different types of fractures which may occur in case of rupture. They are pure ductile fracture, cup and cone fracture and brittle fracture. Conversely, unlike the first two kinds of fractures which occur in ductile materials, the third kind takes place in case of brittle materials. All these three types are explained along with pictures in the following paragraphs.

Pure Ductile Fracture

Figure 9 Pure Ductile Fracture

This kind of fracture occurs in very ductile materials such as soft metals (at room temperature), other metals, polymers, glasses at high temperature. In a pure ductile type of fracture the cross-section of the material at the point of rupture reduces considerably and finally breaks into two parts.

Cup & Cone Fracture

Figure 10 Cup & Cone Fracture

This kind of fracture happens in moderately ductile materials mainly due to uniaxial tension. It is essentially the separation of a body into two separate pieces due to the application of excessive tensile stress. This mode of fracture gets its name from the resulting shapes at the end of the broken pieces after a failure has occurred.

Brittle fracture

Figure 11 Brittle Fracture

This kind of fracture happens only in brittle materials such as pieces of chalks or concrete. During brittle fractures the structure fails with a slicing kind of failure.

Numerical-4

Q. A bar having original gauge length 600m and diameter 16mm is tested under tensile load. The maximum load that is recorded in the test is 95.5KN. At failure the final length of the bar had become 675mm. Find (i) Ultimate Strength of the bar material, (ii) the percentage elongation.

Solution:

Given,

Original length, L_0 = 600 m

Original diameter, d = 16 mm

Load, p = 95.5 KN

Final length, L_f = 675 mm

Then,

Step-1: Find the area of the cross-section

Cross-sectional area of the bar = $\pi \times \frac{d^2}{4} = \frac{22}{7} \times \frac{16^2}{4}$ = 201.14 mm^2 ≈ 201mm^2

Step-2: Calculate the Ultimate Strength

Ultimate strength = $\frac{maximum\ load}{cross\text{−}sectional\ area} = \frac{95.5 \times 10^3}{201}$ = 475.12 N/mm^2 ≈ 475 N/mm^2

Step-3: Compute the Percentage Elongation

Percentage Elongation $= \frac{L_f - L_0}{L_0} \times 100 = \frac{675-600}{600} \times 100 = 12.5$

Therefore,

(i) The ultimate strength of the bar was found to be approximately 475 N/mm^2.

(ii) The percentage elongation of the bar was calculated to be 12.5.

Modular Ratio

Case-1: Two bars of different lengths and cross-sectional diameters

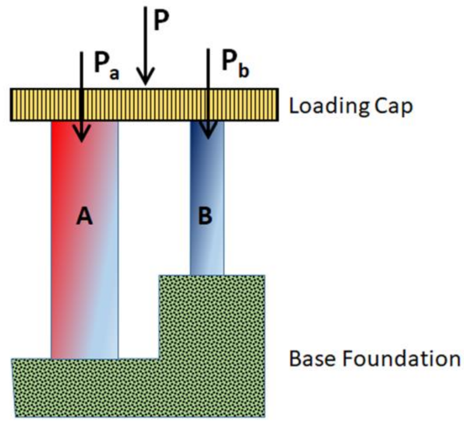


Figure 12 Modular Ratio

Finally, let us go to the concept of modular ratio. So let there be two bars A and B the details of which are given below in the Table 1.

Table 1 Modular Ratio

	Bar-A	Bar-B
Length	L_a	L_b
Cross-sectional Area	A_a	A_b
Young's Modulus	E_a	E_b
Force	P_a	P_b

Let the total load be P such that

$$P = P_a + P_b$$

Now,

$$\text{Stress in bar-A} = \sigma_a = \frac{P_a}{A_a}$$

$$\text{And Stress in bar-B} = \sigma_a = \frac{P_b}{A_b}$$

Then,

$$\text{Elongation in the bar-A} = \Delta L_a = \frac{\sigma_a}{E_a} \times L_a$$

And Elongation in the bar-A = $\Delta L_b = \frac{\sigma_b}{E_b} \times L_b$

But,

$\Delta L_a = \Delta L_b$

i.e., $\frac{\sigma_a}{E_a} \times L_a = \frac{\sigma_b}{E_b} \times L_b$

i.e., $\frac{\sigma_a}{\sigma_b} = \frac{E_a}{E_b} \times \frac{L_b}{L_a}$

i.e., $\frac{\sigma_a}{\sigma_b} = m \times \frac{L_b}{L_a}$

i.e., $\sigma_a = m \times \frac{L_b}{L_a} \times \sigma_b$

Where, m is the modular ratio.

So theoretically,

The Modular Ratio is the ratio of Modulus of Elasticity of two materials.

Generally Modular Ratio is expressed as modulus of stronger material to relatively weaker material. Usually it is used in case of composite beams.

Finally, the force or stresses in each bar can be computed by the following equation:

$$P = P_a + P_b = (\sigma_a \times A_a) + (\sigma_b \times A_b)$$

Case-2: Two bars having same original length

We know that,

$$\sigma_a = m \times \frac{L_b}{L_a} \times \sigma_b$$

i.e., $\frac{\sigma_a}{\sigma_b} = \frac{E_a}{E_b} = m$

i.e., $\sigma_a = m \times \sigma_b$

Again we know that,

$$P = (\sigma_a \times A_a) + (\sigma_b \times A_b)$$

Then,

Substituting $\sigma_a = m \times \sigma_b$

$$P = (m \times \sigma_b \times A_a) + (\sigma_b \times A_b) = \sigma_b (m A_a + A_b)$$

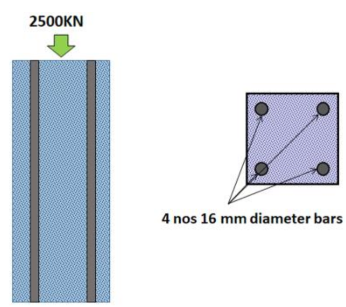
Now,

Substituting $\sigma_b = \frac{m}{\sigma_a}$

$P = (m \times \frac{m}{\sigma_a} \times A_a) + (\frac{m}{\sigma_a} \times A_b) = \frac{m}{\sigma_a}(m\,A_a + A_b)$

Numerical-5

Q. A column having cross-sectional dimension as 300mm × 300mm, is reinforced with four bars of 16 mm diameter. The total load on the column is 2500 KN. Young's Modulus of elasticity of steel is 2×10^5 N/mm^2 and that of concrete is 0.17×10^5 N/mm^2. Find the stress in the steel and column. Also find the load share between concrete and steel.



Solution: Given,

Cross-sectional area of column, $A_c = (300 \times 300)$ mm^2 = 90000 mm^2

Diameter of steel bars = 16 mm

Then, Cross-sectional area of a steel bars = $3.142 \times (\frac{16}{2})^2 = 201.14$ mm^2 ≈ 201mm^2

Then, cross-sectional area of 4 steel bars, A_s = 804 mm^2

Also given,

Total load, P = 2500 KN

Young's Modulus of Elasticity of steel, $E_s = 2 \times 10^5$ N/mm^2

Young's Modulus of Elasticity of concrete, $E_c = 0.17 \times 10^5$ N/mm^2

Now,

Step-1: Find the Stress in concrete

We know that,

$$\frac{\sigma_s}{\sigma_c} = \frac{E_s}{E_c} = \frac{2 \times 10^5}{0.17 \times 10^5} = \frac{2}{0.17} = 11.7$$

i.e., $\sigma_s = 11.7\ \sigma_c$

Again,

We also know that,

$$P = (\sigma_s \times A_s) + (\sigma_c \times A_c)$$

i.e., $P = (11.7\ \sigma_c \times 804) + [\ \sigma_c \times (90000\text{-}804)] = \sigma_c\ (9406.8 + 89{,}196)$

i.e., $P = 98{,}602.8\ \sigma_c$

i.e., $98{,}602.8\ \sigma_c = 2500 \times 10^3$

i.e., $\sigma_c = 25.35\ \text{N/mm}^2$

Step-2: Find the stress in steel

We have, $\sigma_s = 11.7\ \sigma_c = 11.7 \times 25.35 = 296.64\ \text{N/mm}^2$

Step-3: Compute the load share concrete and steel

Load share of Concrete = $25.3 \times (90000\text{-}804) = 2256.66$ KN $\approx$ 2256 KN

Load share of Steel = $296.6 \times 804 = 238.466$ KN $\approx$ 238 KN

Therefore, the strain in steel was found to be 296.64 N/mm^2 and that of concrete was found to be 25.35 N/mm^2. Furthermore, the load share in the former and latter was found to be approximately 238 KN and 2256 KN respectively.

Conclusion

Thus, the following points may be noted in conclusion:

- Elastic material possesses various parametric constants like Poisson's Ratio, Modulus of Rigidity and Bulk Modulus.
- They all are interrelated with the Young's Modulus of Elasticity.
- Modular Ratio is one of the prime factor that governs the design of composite structures.

References

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- Basic Structures for Engineers and Architects By Philip Garrison, Blackwell Publisher
- Understanding Structures: *An Introduction to Structural Analysis* By Meta A. Sozen & T. Ichinose, CRC Press

Homework

Q1. The following data table was obtained during a tensile test of a metal sample. The length of the sample is 1.2 m and the cross-section area is 200 mm^2 . The sample fails into two parts at 80 KN load, mentioned at the last row of the table.

Tensile Load	**Elongation**
KN	**mm**
10	0.4
20	0.8
30	1.2
40	1.6
50	2
60	2.4
70	8
80	12.6
90	18.2
100	34.8
90	50.4
80	75

Find the following:

1. Modulus of Elasticity
2. Yield Stress
3. Ultimate Stress
4. Rapture Stress
5. Permissible working stress keeping factor of safety as 1.5
6. Percentage elongation

Q2. Two posts of different material, cross-section area and length are supporting 50KN load. Find the following referring the figure below:

(i) Modular Ratio and

(ii) Stresses in the posts