Urban Landuse and Transportation Planning
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Lecture - 44
Link Assignment 2

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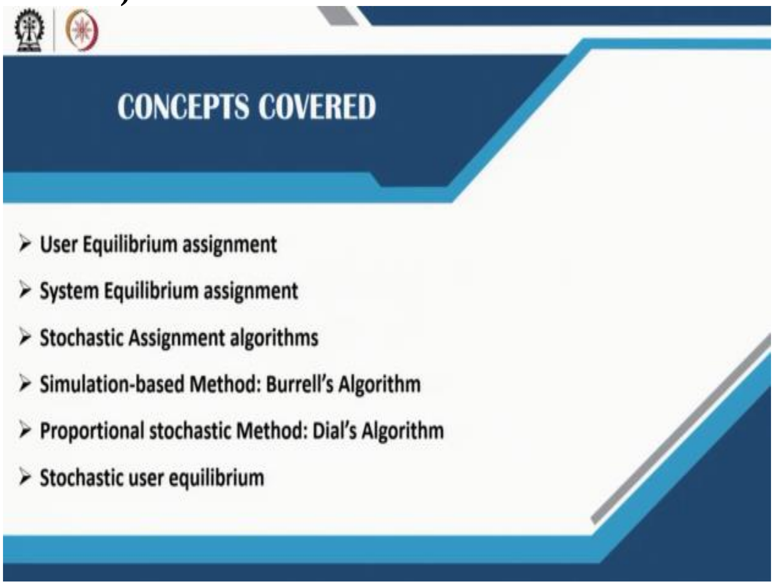

In lecture 44, we will cover the second part of link assignment. In the previous lecture, the deterministic assignment techniques were discussed. In this lecture, other assignment techniques based on Wardrop's Principle will be covered such as, user equilibrium assignment, system equilibrium assignment, and stochastic assignment. Some methods specific to stochastic assignment such as simulation methods like Burrell's algorithm and proportional stochastic method like Dial's algorithm will also be discussed. Besides, the concept behind the stochastic user equilibrium will also be looked at.

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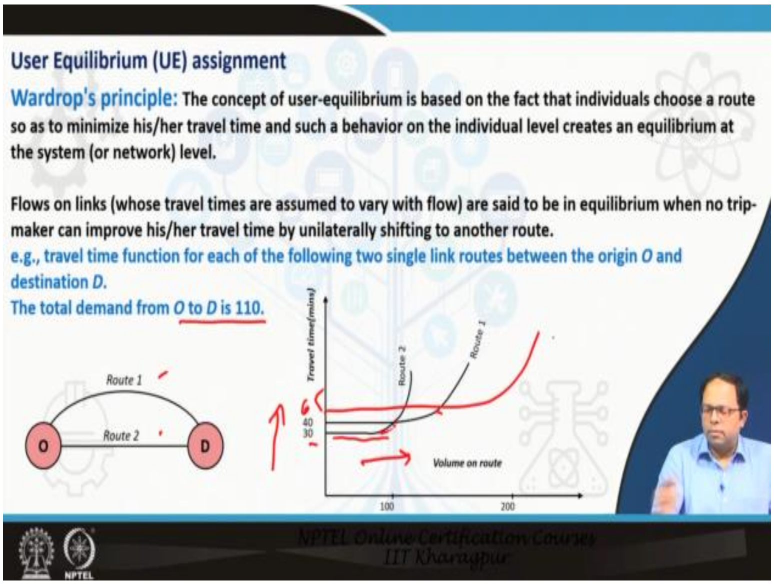

User Equilibrium Assignment

In the earlier lectures, the basic concept related to Wardrop's principle have already been discussed. It states that the individuals choose a route in such a way that his or her travel time is minimized. Such a behavior on the individual level creates an equilibrium at the system or the network level. This was the first principle as stated by Wardrop where everybody tends to choose the shortest route as per their knowledge. It leads to an equilibrium where everybody is divided amongst certain routes. The flows on links whose travel times are assumed to vary with flow are set to be in equilibrium when no trip maker can improve his or her travel time by unilaterally shifting to another route. It implies that the travel times in all the used routes are same at the state of equilibrium. For example, in the diagram provided above there are two routes between O and D and the total demand between O and D is 110. In the graph provided above, travel time along the links is plotted against the link volume. It is observed that the free flow travel time (for link volume =0) in route 1 and route 2 are 40 and 30 mins respectively. Therefore, the second route is faster compared to the first and automatically people will start using the second road. Once the flow in the second route increases, the travel time also increases. When the flow in this route reaches 100 or the total number of trips between O and D, i.e. 110 is assigned to this route, the travel time increases to almost 40. Therefore, the next 10 persons who will be assigned between O and D might find route 1 to be more attractive compared to route 2 in terms of travel time after comparison. So, they will tend to choose route 1 and will continue to do the same till the travel time again increases and route 2 becomes more attractive. So, eventually the travel time in both the routes reach the same level. Any other route with higher travel time is not chosen. Suppose another route with free flow travel time of 65 is added to the network. This route is only chosen if the travel time in route 1 and 2 becomes higher than this new route. If the travel time in the new route remains higher, this route is not chosen at all. Therefore, volume would not be assigned to this route.

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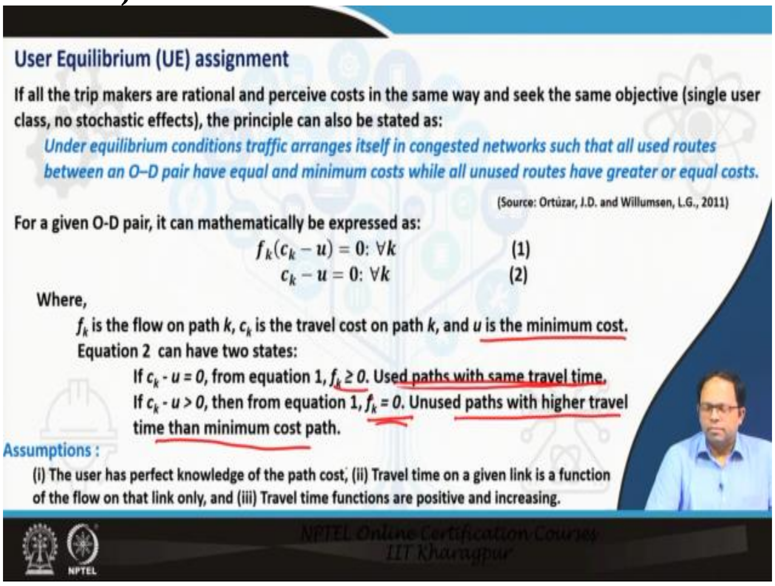

In user equilibrium, all trip makers are rational and perceive cost in the same way and seek the same objective, i.e. to reach the destination using the shortest path. In this regard, we will consider a single user class i.e. the perceptions about cost, time and preferences across the individuals will remain the same. Besides, no stochastic effects are included for this equilibrium which implies that there is no variability among the population. Based on the above assumptions, the principle could be also stated as,

Under equilibrium conditions, traffic arranges itself in congested networks such that all used routes between an OD pair have equal and minimum cost, while all unused routes have greater or equal cost.

Therefore, for a given OD pair, the above principle can be mathematically expressed as,

$$f_k (c_k - u) = 0: \forall k \quad (1)$$

$$c_k - u = 0: \forall k \quad (2)$$

where, f_k is the flow on path k, c_k is the travel cost on path k and u is the minimum cost of travel for that path. The second equation can assume two states,

1) If c_k - u = 0, from equation 1, $f_k \geq 0$. In this case, used paths will have the same travel time.

2) If c_k - u > 0, then from equation 1, $f_k = 0$. In this case, unused paths will have higher travel time than the minimum cost path. Therefore, if the travel time in path k is higher than the minimum cost path, then flow on path k is 0. On the other hand, if the travel time is

similar to the minimum cost along the paths, the flow will be assigned to these paths i.e. these paths will be used. Some assumptions are made to conduct assignment under user equilibrium. Firstly, we assume that the user has perfect knowledge of the path cost and the users have similar preferences. Secondly, the travel time on a given link is a function of the flow on that link only, i.e., the travel time is affected by the total volume in that link and not by the volume of other links. The final assumption is that the travel time functions are positive and increasing.

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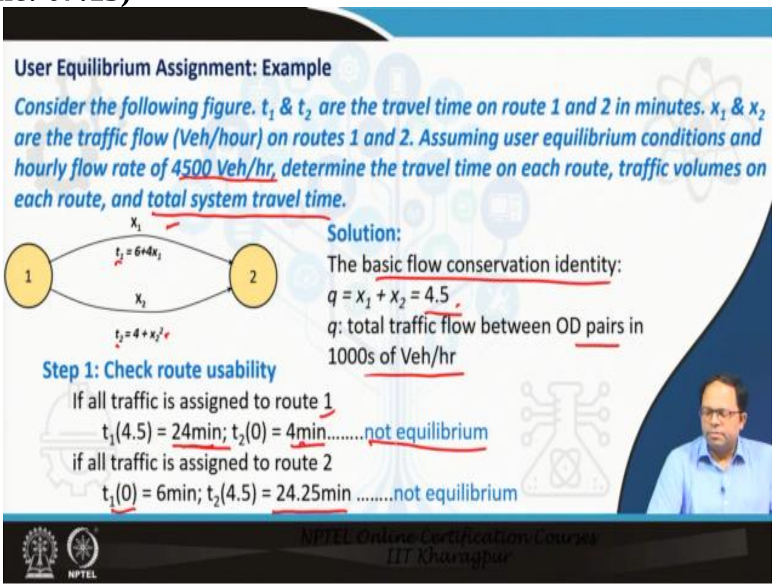

Let us consider the following figure. t_1 & t_2 are the travel time on route 1 and 2 in minutes respectively. x_1 & x_2 are the traffic flow (Veh/hour) on routes 1 and 2 respectively. Assuming user equilibrium conditions and hourly flow rate of 4500 Veh/hr, we need to determine the travel time on each route, traffic volumes on each route, and total system traffic. q is the total traffic volume in 1000s veh/hr between the origin 1 and destination 2. Therefore, the total flow between the above pair is 4.5. Based on the basic flow conservation identity, the total traffic flow is represented as, $q = x_1 + x_2 = 4.5$. Now, let us check whether equilibrium conditions are satisfied by assigning the traffic in one of the routes. Let us assign the entire traffic volume to route 1. Now, t_1 is given by the following equation,

$$t_1 = 6+4x_1$$

Substituting x_1 with 4.5 in the above equation, the travel time for route 1 is estimated as 24 minutes. Now, t_2 is given by the following equation,

$$t_2 = 4 + x_2^2$$

Since the entire volume is assigned to route 1, no flow will be assigned to route 2. Therefore, x_2 is taken as 0. Based on this value of x_2, t_2 is estimated as 4 minutes. While the travel time in route 1 is a linear function, the travel time in route 2 is a quadratic function. Therefore, when the flow increases in the route 2, there is drastic increase in travel time. Since, t_1 & t_2 are not equal, the equilibrium condition is not attained. Similarly, if the entire traffic is assigned to route 2, t_2 is estimated as 24.25 minutes and the travel time in route 1 is estimated as 6 minutes. So, in this scenario also the equilibrium conditions are not satisfied.

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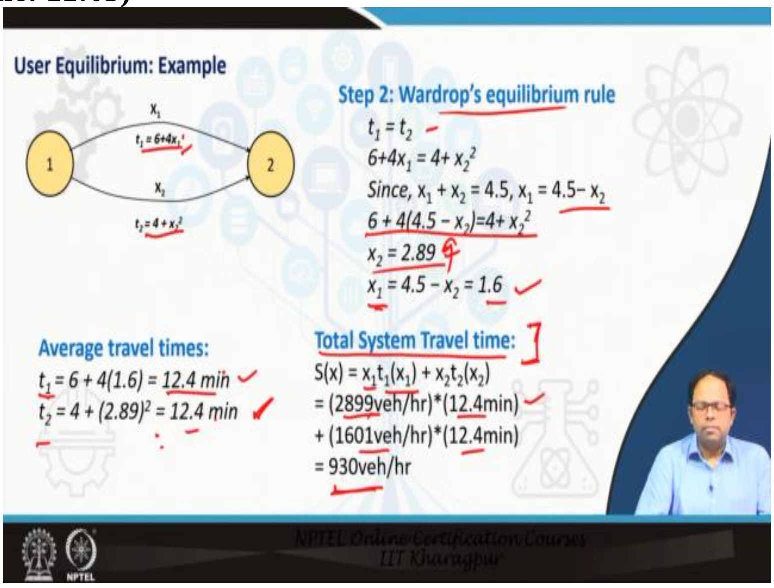

Now, let us assume that based on the Wardrop's equilibrium rule,

$$t_1 = t_2$$

Substituting t_1 and t_2 with the equations provided above we get,

$$6+4x_1 = 4+ x_2^2$$

Now, based on the flow conservation identity we have,

$$x_1 + x_2 = 4.5$$

So, x_1 can be expressed in terms of x_2 as,

$$x_1 = 4.5 - x_2$$

Substituting the value of x_1 we get,

$$6 + 4(4.5 - x_2)=4+ x_2^2$$

Solving the above equation, the value of x_2 is obtained as 2.89 whereas x_1 is estimated as (4.5 - 2.8) i.e. 1.6. So, these values can be inserted in the travel time equation to determine the average travel time in both the links. The average travel time in both the links must be equal since the flows are estimated for equilibrium condition. The average travel time in both the links is found to be 12.4 minutes. So, this is the equilibrium time or the travel time in both the routes under equilibrium. Now, the total flow in x_1 is 1601 veh/hr and the total flow in x_2 is 2899 veh/hr. Now, the total system traffic can be estimated using the following equation,

$$S(x) = x_1 t_1(x_1) + x_2 t_2(x_2)$$

Where, x_1 and x_2 are the flows under equilibrium in link 1 and link 2 respectively. $t_1(x_1)$ and $t_2(x_2)$ are travel time at x_1 and x_2 respectively. Based on the values obtained above for the above parameters, the total system traffic under equilibrium is estimated as 930 vehicles. Therefore, the system can handle an average of 930 vehicle trips between the given OD pair under equilibrium. This can measure efficiency of a given system.

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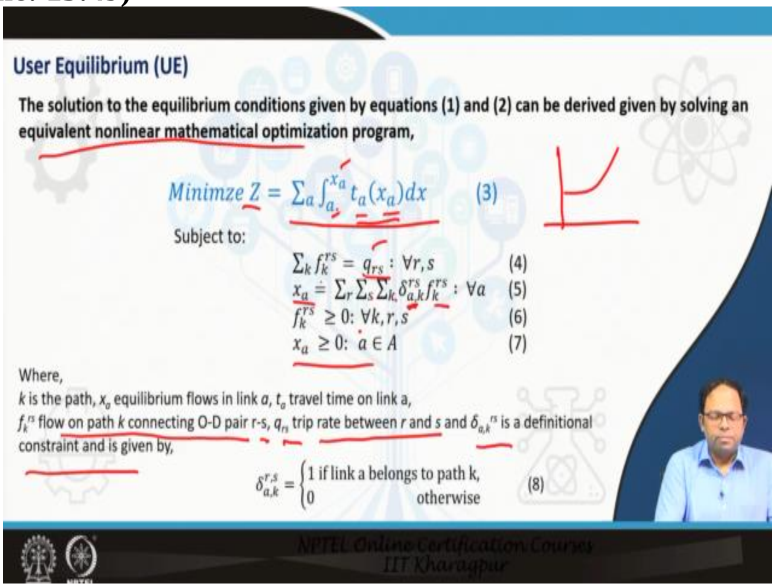

The first principle of Wardrop's equilibrium can be solved mathematically. A single route is composed of multiple links. So, we need to determine the total area under each of the flows along these links. In order to achieve equilibrium, the total area must be minimum since the shortest route must be determined between an OD pair as per the Wardrop's equilibrium. This can be solved using an equivalent nonlinear mathematical optimization program, which is expressed as follows:

$$Minimze\ Z = \sum_a \int_a^{x_a} t_a(x_a)dx$$

which is subjected to the following constraints:

$$\sum_k f_k^{rs} = q_{rs} : \forall r,s$$

$$x_a = \sum_r \sum_s \sum_k \delta_{a,k}^{rs} f_k^{rs} : \forall a$$

$$f_k^{rs} \geq 0: \forall k,r,s$$

$$x_a \geq 0: a \in A$$

where f_k^{rs} is the flow on link k between the OD pair r and s, x_a is the equilibrium flows in path a, t_a is the travel time on link a, q_{rs} is the trip rate between r and s and δ_{rs} is a definitional binary constraint, i.e. 1 and 0 based on the inclusion of a link within a path. If the flow is assigned to a link, then that flow will only be considered for equilibrium flow if the link belongs to the shortest path. Therefore, the equilibrium flow along a path is estimated by considering the total flows on the links included in that path. Besides, f_k^{rs} in a link k must always be greater than 0. Also, the equilibrium flows along the links must be greater than 0. Therefore, the travel time between an OD pair must be minimized by satisfying all the above conditions.

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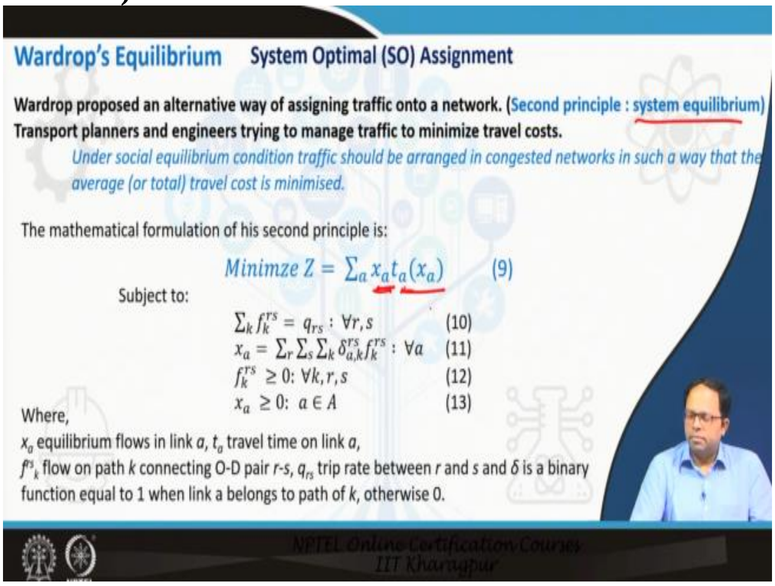

System Equilibrium Assignment

The extension of the first principle can be observed in Wardrop's second principle which talks about the system optimal assignment. The second principle, also known as system equilibrium, proposes an alternative way of assigning traffic onto the network. In this equilibrium, the transport planners and engineers who are trying to minimize traffic or manage traffic aim to minimize travel cost over an entire city. A desired minimum travel time in different corridors in the network can be achieved through different strategies like route extension, right-of-way extension, new route addition, signal management and more. The second principle can be stated as,

Under social equilibrium condition traffic should be arranged in congested networks in such a way that the average (or total) travel cost is minimised.

Therefore, the total volume as well as the travel time in each link must be minimized. Mathematically, the above principle can be expressed as,

$$Minimze\ Z = \sum_a x_a t_a(x_a)$$

Where, x_a is the total volume in each link, $t_a(x_a)$ is the travel time of each link for flow x_a . The above optimization problem is subjected to the following constraints:

$$\sum_k f_k^{rs} = q_{rs} : \forall r,s$$

$$x_a = \sum_r \sum_s \sum_k \delta_{a,k}^{rs} f_k^{rs} : \forall a$$

$$f_k^{rs} \geq 0 : \forall k,r,s$$

$$x_a \geq 0 : a \in A$$

Where, f^{rs}_k is flow on path k connecting O-D pair r-s, q_{rs} is trip rate between r and s and δ is a binary function equal to 1 when link a belongs to path of k, otherwise 0. While the travel time is minimised under user equilibrium, the overall system cost is minimised for system equilibrium. Therefore, both the link volume and the travel time is considered in the overall cost of the system. Moreover, in this equilibrium process, we do not consider the travel time minimization of each individual. Therefore, some user may have higher travel time compared to others, but the overall system travel time will be minimized.

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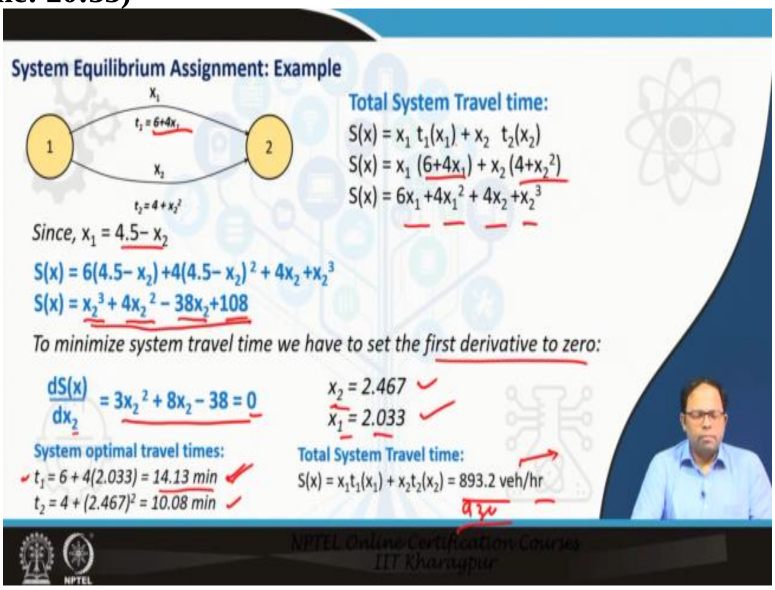

Let us consider the previous example to explain the system equilibrium. The total system travel cost can be represented as,

$$S(x) = x_1 * t_1(x_1) + x_2 * t_2(x_2)$$

Based on the functions of t_1 & t_2 provided above, the system travel cost function can be written as,

$$S(x) = x_1 * (6+4x_1) + x_2 * (4+x_2^2)$$

$$S(x) = 6x_1 + 4x_1^2 + 4x_2 + x_2^3$$

Substituting the value of x_1 the equation takes the following form,

$$S(x) = x_2^3 + 4x_2^2 - 38x_2 + 108$$

So, to minimize system travel cost, we have to set the first derivative to 0. Therefore, the system travel cost is differentiated with respect to x_2. The following expression is derived:

$$\frac{dS(x)}{dx} = 3x_2^2 + 8x_2 - 38 = 0$$

If we solve the above equation, x_2 is estimated as 2.467 and the value of x_1 is 2.033. When the travel time functions are replaced with the values calculated above, t_1 becomes 14.13 and t_2 becomes 10.08 minutes. The travel times so estimated are completely different from the one we estimated earlier. For the user equilibrium, the travel time in both the links was found to be similar, i.e. 12.4 minutes. Here, the travel times are different. The total system travel cost based on the given equation is estimated as 893.2 vehicles which is less than the cost determined for user equilibrium assignment (930 vehicles). Although the minimum cost is achieved over the system, all people will not be travelling along the shortest route. Some people will be using a longer route. Therefore, the overall system is more stable with a total minimum cost. The transport planners will focus on the reduction of the overall cost of the system. So, probably for them, this is an ideal method which can be employed.

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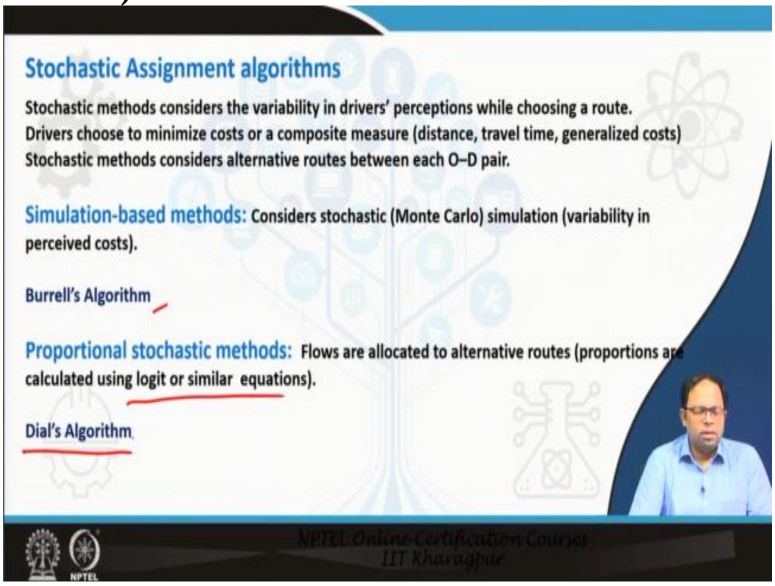

Stochastic Assignment

In the equilibrium assignment, the variation in perception of each individual is not included. The stochastic assignment algorithms consider human behavior to conduct assignment for a given network. The stochastic methods consider variability in drivers' perception while choosing a route. The drivers choose to minimize cost or a composite measure (distance, travel time or generalized cost). So, the drivers can consider any one of the parameters or a combination of some of them. The stochastic method considers alternative routes between each OD pair based on the knowledge of the drivers or the perception of the drivers about the shortest path. This will lead to a certain form of assignment based on the driver's route choice. Stochastic assignment can be carried out by using two common methods, simulation-based method, and proportional stochastic methods.

Simulation-based method – this method considers stochastic (for e.g.- Monte-Carlo) simulation to incorporate the variability in the perceived cost. An example of the same is Burrell's algorithm.

Proportional stochastic methods – this method considers that the flows are allocated to alternative routes. The proportions are calculated using logit or similar equations. The cost of a route is considered to be an exponential function. Dial's algorithm is an example of this method.

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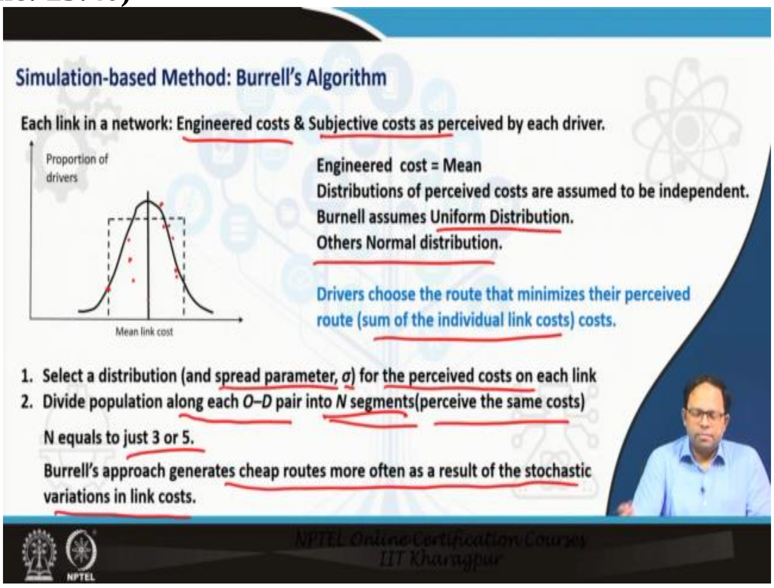

Simulation-based Method

Burrell's algorithm, a simulation-based method, considers engineered cost and subjective costs as perceived by each driver for each link in a network. Such consideration takes care of the variation in perception. The mean is considered as the engineered cost. The distribution of perceived cost is assumed to be independent. This is because of perceptions regarding cost vary across drivers. While this algorithm assumes that this variation follows a uniform distribution, some other algorithms assumes that the perceived cost is normally distributed. This distribution is basically the distribution of the observations of perceptions of individuals regarding link cost. The drivers choose the route that minimizes their perceived route costs, which is the sum of individual link costs. Therefore, the drivers will choose the route that will yield the least cost based on their perceptions. In order to represent the perceived cost on each link, a distribution must be selected. Besides, the spread parameter for this distribution, σ must also be decided. Based on the distribution and spread parameter, we can generate random numbers for individuals to represent the cost a person actually perceives for a link. Each individual person has to select costs for each link which will lead to different shortest path alternatives. If the process is carried out for every individual, then it might lead to complexities in computation. Now, there will be a group of individuals for whom the perceived costs for a link will be similar. Therefore, the population along each OD pair can be divided into N segments, as observed in incremental methods. These segments are based on these group of people who will assume the same cost for this link. In most cases, N is just equal to 3 or 5. So, there are maximum 3 or 5 groups with different perceptions of route cost. The Burrell's algorithm generates cheap routes more often as a result of stochastic variation in the link cost because a distribution is assumed for the link costs. The random numbers, so generated will tend to replicate the actual link cost since these numbers will be close towards the mean. Such characteristics of the distribution will automatically lead to cheaper routes in most of the cases.

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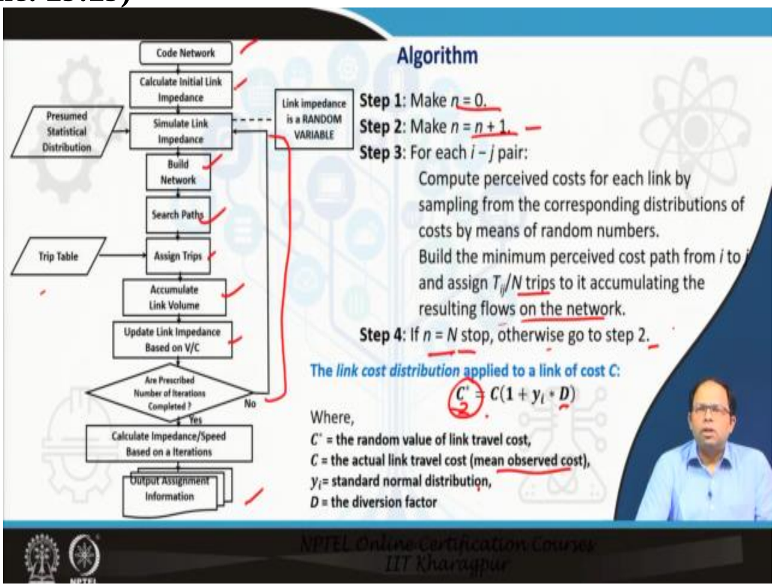

Let us now understand the process behind the algorithm. The first step is to code the network in which the different characteristics of the network are assigned. Next, we need to calculate the initial link impedances using a statistical distribution because link impedance is a random variable. The distribution is assumed based on a known mean value of the link impedance. Then we determine random numbers as link impedances through simulation. Based on the simulated values of link impedances, we can build a network. Then appropriate paths are searched in that network. The trip rates or the number of trips between an OD pair can be identified from the trip table. Based on the given table, we can assign trips for group of individuals into the paths which have been determined in the previous step. Based on the assigned trips, the link volume can be accumulated, and the link impedances are updated based on volume by capacity ratio. Once the updated link impedances are obtained, we must check whether prescribed number of iterations are completed. If the iteration is yet to reach the prescribed number, the link impedances are simulated, and the above process is repeated. Once the prescribed number of iterations is reached, the impedance must be calculated based on speed and the assignment information can be obtained as an output. So, the link cost distribution applied to a link of cost *C* can be expressed as,

$$\boldsymbol{C}^{*} = \boldsymbol{C}\left(\boldsymbol{1} + \boldsymbol{y}_{\boldsymbol{i}} * \boldsymbol{D}\right)$$

where C^* is the random value of link travel cost or the random cost, *C* is the actual link travel costs (mean of the observed costs). And y_i is the standard normal distribution, and *D* is the diversion factor that we assume to give us a random value. The steps involved in the Burrell's algorithm can be summarized as follows:

Step 1: Make $n = 0$.

Step 2: Make $n = n + 1$.

Step 3: For each $i - j$ pair:

Compute perceived costs for each link by sampling from the corresponding distributions of costs by means of random numbers.

Build the minimum perceived cost path from *i* to *j* and assign T_{ij}/N trips to it accumulating the resulting flows on the network.

Step 4: If n = N stop, otherwise go to step 2.

Therefore, the shortest paths are created out of the random values generated for link costs and a group of people assumed to have similar perception is assigned to that shortest path. Once the assignment is conducted, the entire traffic volume on the network is updated. Once the algorithm terminates after a fixed number of iteration, the total trips between OD pairs based on the trip table has been assigned to the network.

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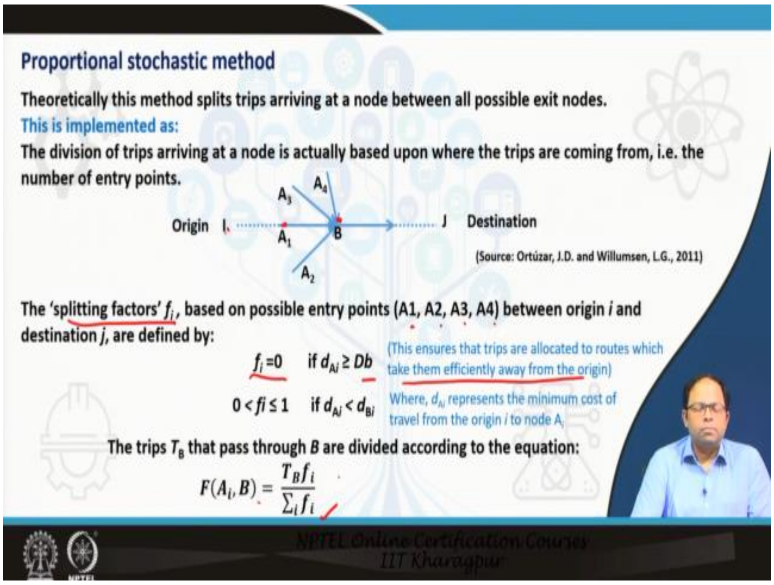

Proportional Stochastic Method

Theoretically, the proportional stochastic method splits the trips arriving at a node between all possible exit nodes. The number of trips going from one node to another are divided. Therefore, the trips are split into different pathways using this method. But the implementation of this method is based on the entry points of the trips. Thus, the division of trip arriving at a node is actually based upon where the trips are coming from, that is the number of entry points. For example, in the above diagram, there is a point B between origin *i* and destination *j*. The trips are arriving from A_1, A_2, A_3, and A_4. So, the total number of trips will be divided based on a parameter, known as splitting factor, f_i on the possible entry nodes A_1, A_2, A_3, and A_4 between origin *i* and destination *j*. This splitting factor can be defined as,

$$f_i = 0 \qquad \text{if } d_{Ai} \geq d_{Bi}$$

$$0 < f_i \leq 1 \qquad \text{if } d_{Ai} < d_{Bi}$$

d_{Ai} represents the minimum cost of travel from the origin *i* to node A_i. If d_{Ai} is greater than d_{Bi}, then d_{Bi} is the shortest route and splitting of trips is not required. This ensures that the trips are allocated to routes which take them efficiently away from the origin. But if d_{Ai} is lesser than d_{Bi}, it implies that d_{Ai} is actually the shortest path. Therefore, an alternate path must be adopted to arrive at B through A_i. Thus, a splitting factor between 0 to 1 must be considered for the division in the trips. So, the trips T_B that passes through B must be a divided according to the following equation if $d_{Ai} < d_{Bi}$:

$$F(A_i, B) = \frac{T_B f_i}{\sum_i f_i}$$

Therefore, the entire trip between an OD pair can be proportionately divided into different routes if alternate minimum link cost options are available.

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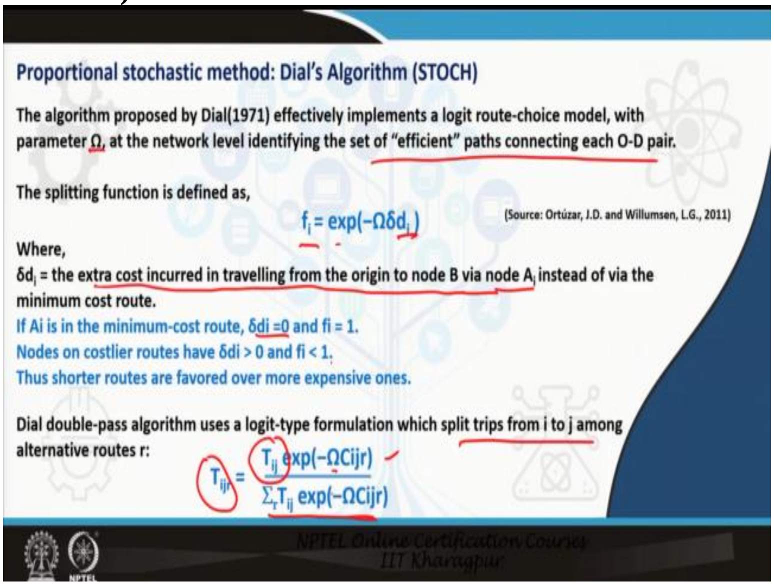

Dial's algorithm, named after its proponent Dial, was proposed in 1971. This algorithm effectively implements a logit route choice model with parameter Ω, at the network level identifying a set of "efficient" paths connecting each OD pair. The selection of an appropriate route from several route options can be represented through discrete functions. Therefore, route choices can be modeled using logistic functions. The splitting function is defined as,

$$\mathbf{f_i = \exp(-\Omega\delta d_i)}$$

where δd_i is the extra cost incurred in travelling from origin to node B via node A_i instead of via the minimum cost route. This extra cost will only be incurred if node A_i is not a part of the shortest path between origin and node B as discussed earlier. Therefore, the cost is estimated proportionately for different route alternatives. So, based on that extra cost, the total trip volume is divided into different route options in a proportionate manner. If A_i is in the minimum cost route, then $\delta d_i = 0$ and $f_i = 1$. In other words, no extra cost will be incurred if node A_i is part of the shortest path and the division of trips will not be necessary. The nodes on costlier routes have δd_i greater than 0 and f_i lesser than 1. It implies that if there is extra cost, then the trips will be proportionately divided among multiple routes. Therefore, shorter routes are favored over more expensive ones because expensive routes will lead to multiple routes. So, Dial proposed a double pass algorithm using a logit type formulation which split trips from *i* to *j* among alternative routes *r*. The proportion of trips for route *r*, T_{ijr} can be mathematically expressed as,

$$\boldsymbol{T_{ijr}} = \frac{\boldsymbol{T_{ij}e^{-\Omega\delta d_i}}}{\sum_r \boldsymbol{T_{ij}e^{-\Omega\delta d_i}}}$$

where T_{ij} is the total amount of travel between *i* to *j*. So, the splitting factor between i and j is expressed as the exponential function of the cost incurred to take the deviation if the intermediate node is not included within a minimum cost path. The proportion of the entire trips between *i* and *j* for each route is determined based on the extra cost incurred for traveling on that route. Similar formulations were observed in mode choice models as well.

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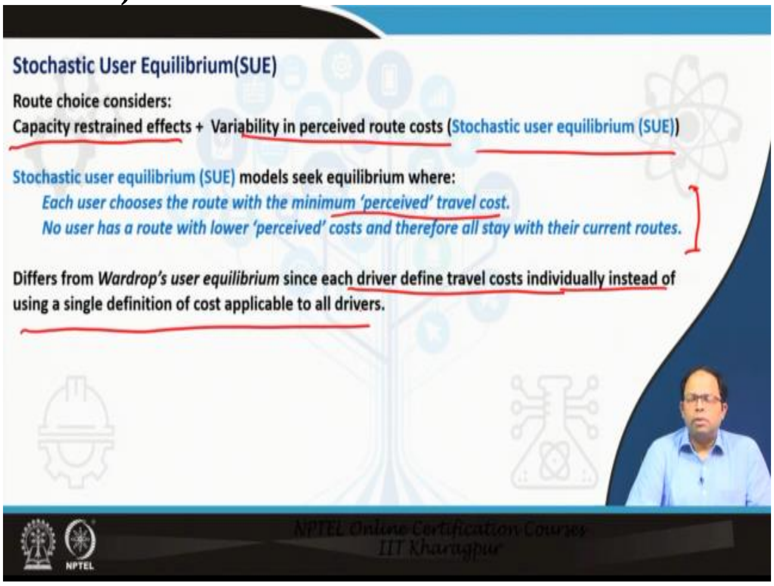

Stochastic User Equilibrium

The stochastic user equilibrium is a modification of the user equilibrium. The capacity restrained effects in terms of travel impedance is included in user equilibrium. The route choice under SUE equilibrium is based on capacity restrained effects as well as the variability in perceived route costs. While capacity restrained effect is deterministic in nature, the stochastic effects are considered through perceived route costs. So, stochastic user equilibrium models seek equilibrium where each user chooses the route with the minimum perceived travel cost and no user has a route with lower perceived cost and therefore, all stay with their current routes. Unlike Wardrop's first principle, stochastic user equilibrium considers the difference between drivers' perception. While the user equilibrium is devoid of behavioral assumptions, stochastic user equilibrium incorporates the human behaviour. This algorithm differs from Wardrop's user equilibrium since each driver define travel cost individually instead of using a single definition of cost applicable to all drivers.

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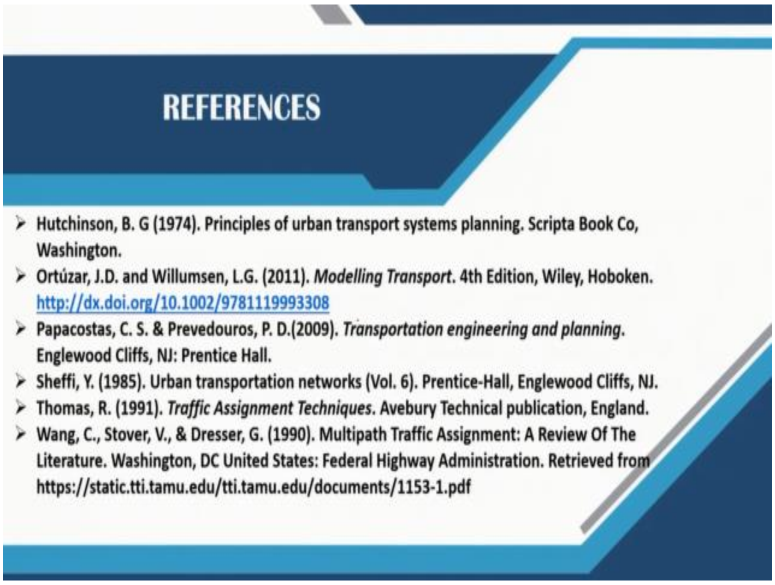

So, these are some of the references you can study.

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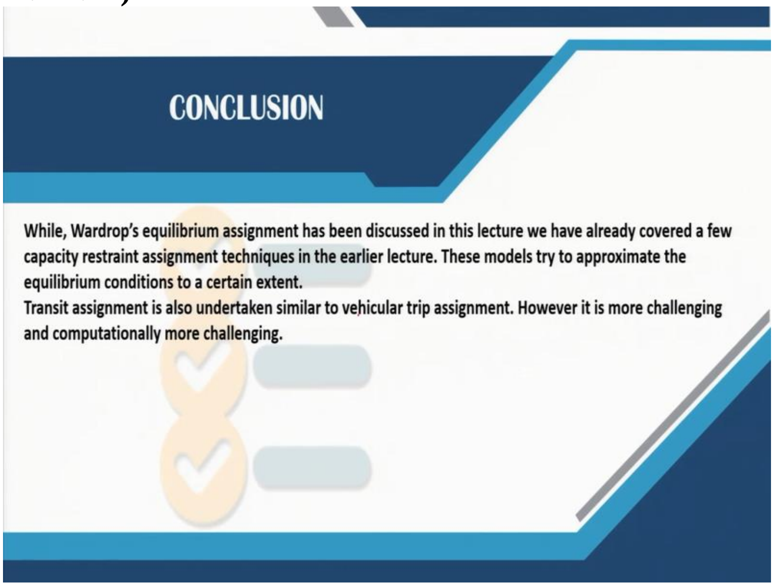

Conclusion

While Wardrop's equilibrium assignment has been discussed in this lecture, a few capacity restraint assignment techniques have already been covered in the earlier lecture. These models try to approximate the equilibrium conditions to a certain extent. Once the vehicular assignment has been conducted along the different routes using the techniques discussed here, some advanced assignment techniques related to stochastic user equilibrium can also be explored. We also need to carry out transit assignment which is also undertaken similar to vehicular trip assignment. However, transit assignment is more challenging computationally since the capacity is provided not by a link but is dependent on number of transit vehicles that are moving along the route. Besides, there are a lot of real-time variations along those routes. Moreover, in a transit network, people can have difficulty in choosing transit routes between two alternatives since these transit routes may overlap each other. The costs associated with transit are also different. While travelling in transit, the individuals consider different cost parameters like number of transfers, fare, the in-vehicle travel time, the out-of-vehicle travel time and many more. Therefore, apart from vehicular assignment as covered in this lecture, techniques related to transit assignment must also be explored. Thank you.