Course Name: Watershed Hydrology

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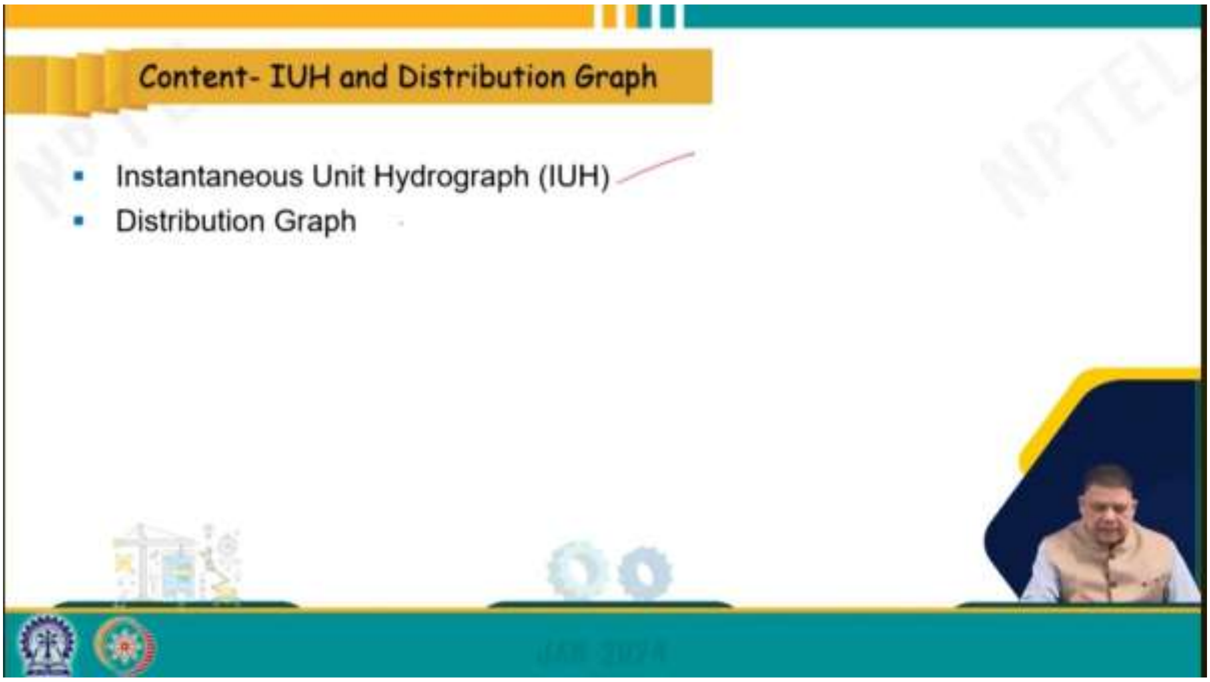
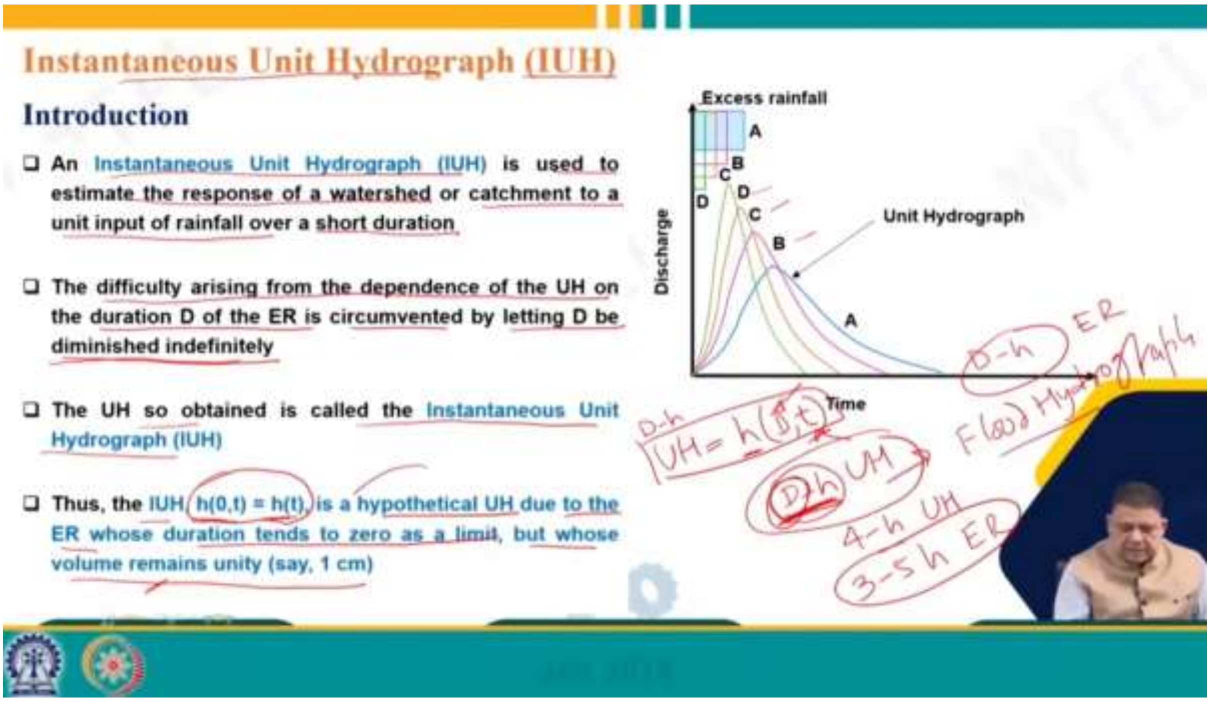
Week: 06

Lecture 28: IUH and Distribution Graph

Hello, friends! Welcome back to this online certification course on Watershed Hydrology. I am Rajendra Singh, a Department of Agriculture and Food Engineering professor at the Indian Institute of Technology Kharagpur. We are in Module 6; this is Lecture number 3 where we will discuss IUH and the distribution graph.



So, the contents of this lecture include an instantaneous unit hydrograph for IUH and a distribution graph.



Starting with the instantaneous unit hydrograph (IUH), an instantaneous unit hydrograph is used to estimate the response of a watershed or catchment to an end-unit input of rainfall over a short duration. Short is infinitesimally small duration and the purpose is that the difficulty arising from the dependence of the UH on the duration D of the effective rainfall is circumvented by letting D be diminished indefinitely.

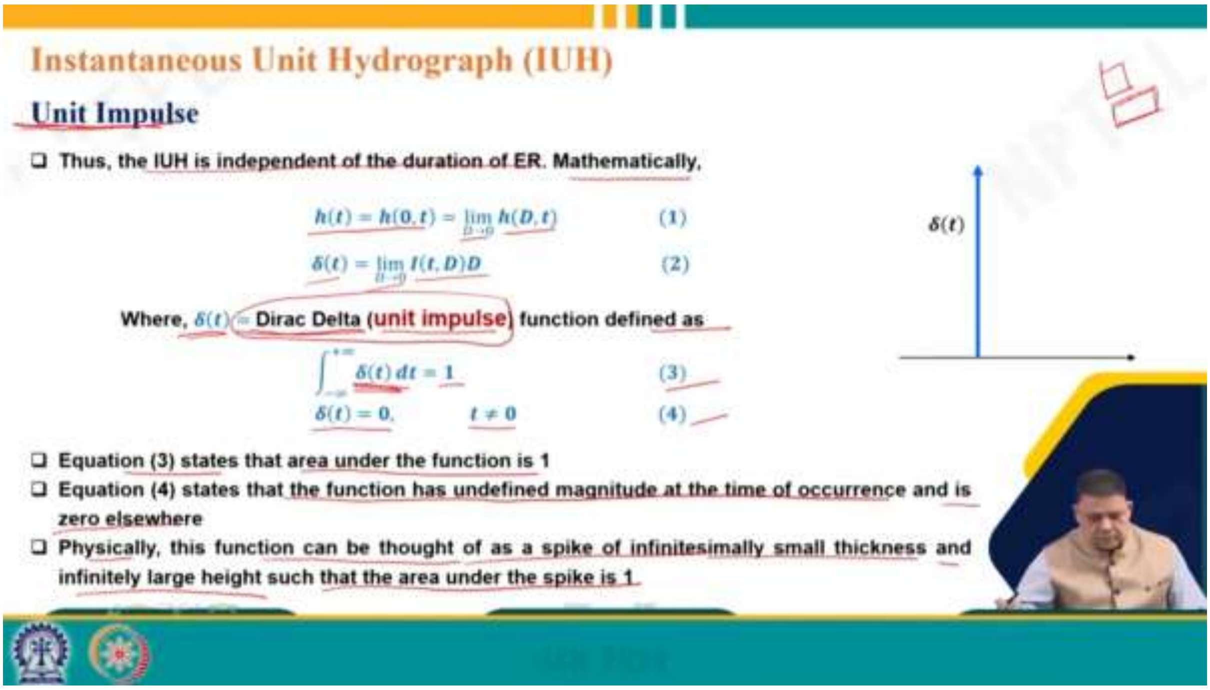
Now, we discussed that if we have a D-hour UH, we can develop a flood hydrograph or direct runoff hydrograph. And then that is by using a D-hour ER, that is effective rainfall must be of the same duration for which we have a unit hydrograph. We also discussed that because in practical field conditions, it is difficult to get data on the corresponding period. So, a relaxation of 25 percent in the UH time is given.

That means, if you have a 4-hour unit hydrograph then you can use it for 3 to 5 hours of effective rainfall of unit duration and magnitude. But suppose we do not meet this requirement also, this relaxation we are not able to manage, then it will not be possible to use unit DH, D-hour unit hydrograph, or 4-hour unit hydrograph to develop the flood hydrograph. So, that means, unit hydrograph theory collapses in that case. So, the idea of the instantaneous unit hydrograph is to make this unit hydrograph independent of D. That is if D becomes infinitesimally small, then the dependence on this D will go away, and that is the concept of the instantaneous unit hydrograph. The UH so derived which is called the instantaneous unit hydrograph. That is if D is diminished completely, then it is an instantaneous unit hydrograph.

So, as you can see this is the unit hydrograph, and this is the excess rainfall duration, it is being shown here A. So, this is the unit hydrograph; if it is reduced, then this is the unit hydrograph; it is further reduced to C, then it becomes this is unit hydrograph D, that is, this is effective rainfall, the magnitude remains the same. So, as you can see, that is the duration is getting short, that is, the magnitude is the vertical length is increasing. So, because the unit area under the curve should be one, ER should be one, and that should be maintained. So, as you can see.

If we continue this process so E F G H such that the timing, that is, this scale becomes infinitesimally small, but still, it will produce the unit hydrograph then that kind of unit hydrograph will be called IUH or instantaneous unit hydrograph. So, basically for unit hydrograph, mathematically when we represent its ordinate, we write h (D, t), that is for D-hour unit hydrograph that is h, that is the ordinate of D-hour unit hydrograph at time t. So, that is how we represent the D-hour unit hydrograph. So, for IUH that ordinate will be h (0, t) because D has reached the 0 value and then it is H(t). So, the ordinate of IUH is h (0, t) or h(t) is the hypothetical unit hydrograph due to the ER whose duration tends to 0 as a limit, but whose volume remains unity say 1 centimetre.

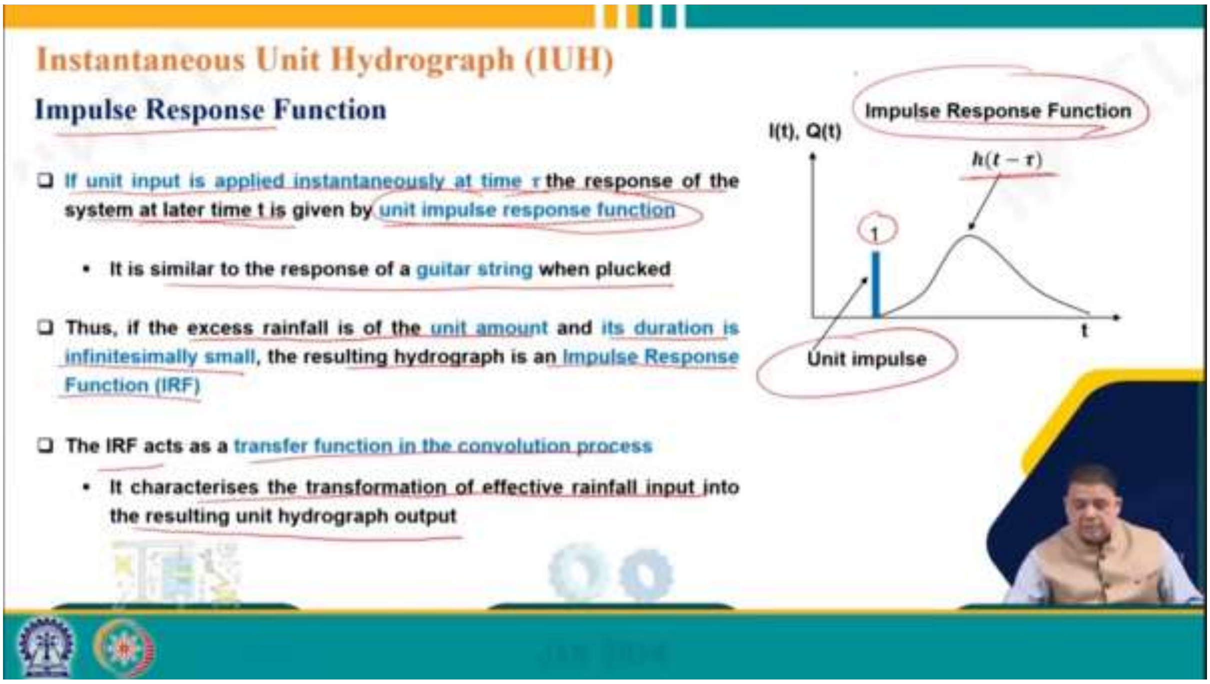
That means it is a unit hydrograph having 1 unit depth of effective rainfall, but the duration tends to 0 as a limit. So, it is infinitely small, not very close to 0, but its volume must remain 1 centimeter. And that is how these ordinates will be called h(0,t) or h(t). Thus, since it is not possible that a unit depth of active rainfall will occur in an infinitely small time or almost instantaneously or at 0 very close to 0 times, it is more of a hypothetical concept than practical, but mathematically, in mathematical operations, there are several applications, and that is why the concept of instantaneous unit hydrograph has been developed.



Now, the IUH is independent of the duration of ER, and mathematically, it can be said as h(t)=h(0,t) That is what we said, that it is the ordinates of the unit hydrograph as the $\lim_{D \to 0} h(D,t)$, where h(D,t) is the unit hydrograph ordinate at time t, if D tending to 0, it becomes h(0,t) or h(t), and δ(t)=$\lim_{D \to 0} I(t,D)D$, where δ(t) basically is called the dirac delta function or the unit impulse function, which is defined by these two equations:$\int_{-\infty}^{+\infty} \delta(t)dt=1$ (equation 3)

δ(t)= 0, t≠0 (equation 4)

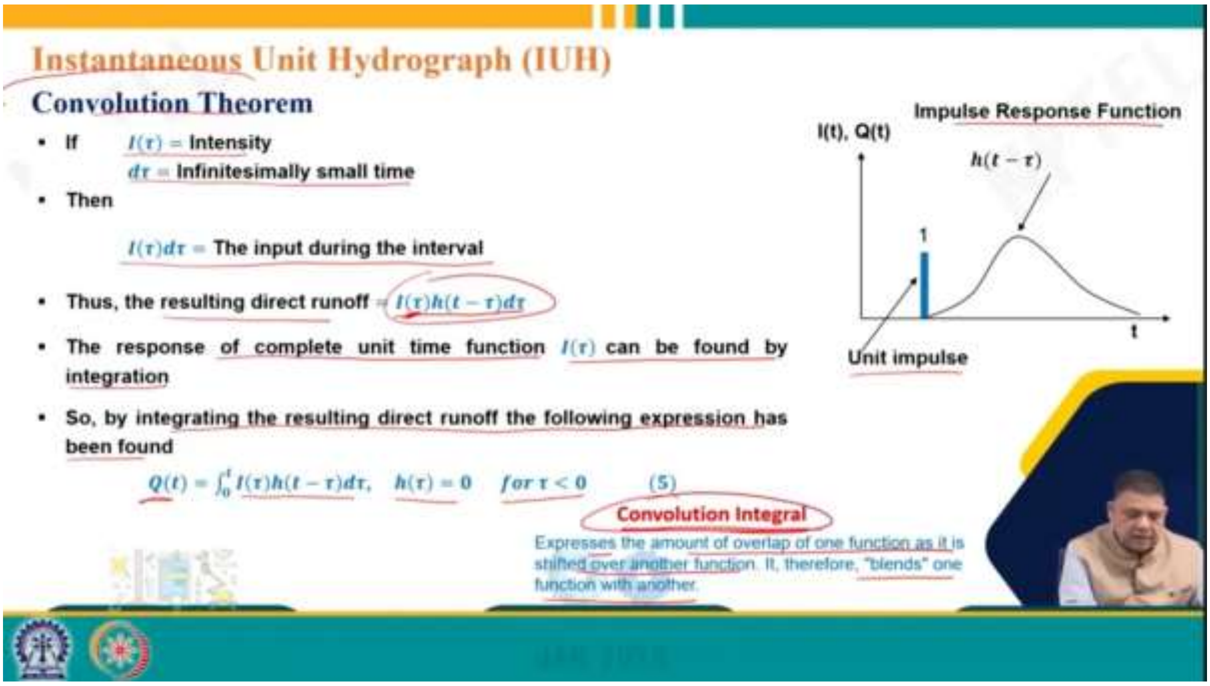
So, equation 3 states that the area under the function is 1, which is true for any effective rainfall. So, it is a unit impulse when we are saying it is representing effective rainfall. So, we all know that for any duration we take, whether D is equal to 2 hours, D is equal to 1 hour, D is equal to 4 hours, there is always 1 unit depth. So, that is why the area under the function is 1. Equation 4 states that the function has an undefined magnitude at the time of occurrence and it is 0 elsewhere. So, it occurs instantly and then it is 0 everywhere else. So, it is a 4-hour period; it is a long period, but here it just occurs instantly. Physically, this function can be thought of as a spike of infinitesimally small thickness. So, its thickness is very small and infinitely large height such that the area under the spike is 1.



Then, on the other side, we have the impulse response function. If a unit input is applied instantaneously at time τ, the response of the system later t is given by the unit impulse response function. As you can see here, this is the unit impulse. Just as we discussed the unit impulse, is infinitely small in thickness but large in height, having a magnitude of 1 that is being applied at a particular moment. Then, its response to the system later t can be given by or traced like this, which is referred to as a unit impulse response function. So, this is an impulse response function, and as you can see here, it is like the response of a guitar string when plucked. So, if you have played the guitar or if you have just tried plucking the guitar string, then you will find that a wave-like thing moves, and you can feel its effect even at the lower end.

That's right. If you pluck it here, then the effect will be felt in different magnitudes as you move down the wire. The same thing is here. Thus, if the excess rainfall is of a unit amount and its duration is infinitesimally small, that is, a unit impulse, the resulting hydrograph is an impulse response function (IRF). So, this is what the impulse response function output looks like, and these are the ordinates here.

The IRF x is a transfer function of the convolution process, that is, it characterizes the transformation of $F(t)$ rainfall input into the resulting unit hydrograph output. So, we have input and then we have output. Input is in the form of a unit impulse and the output is in the form of an impulse response function.

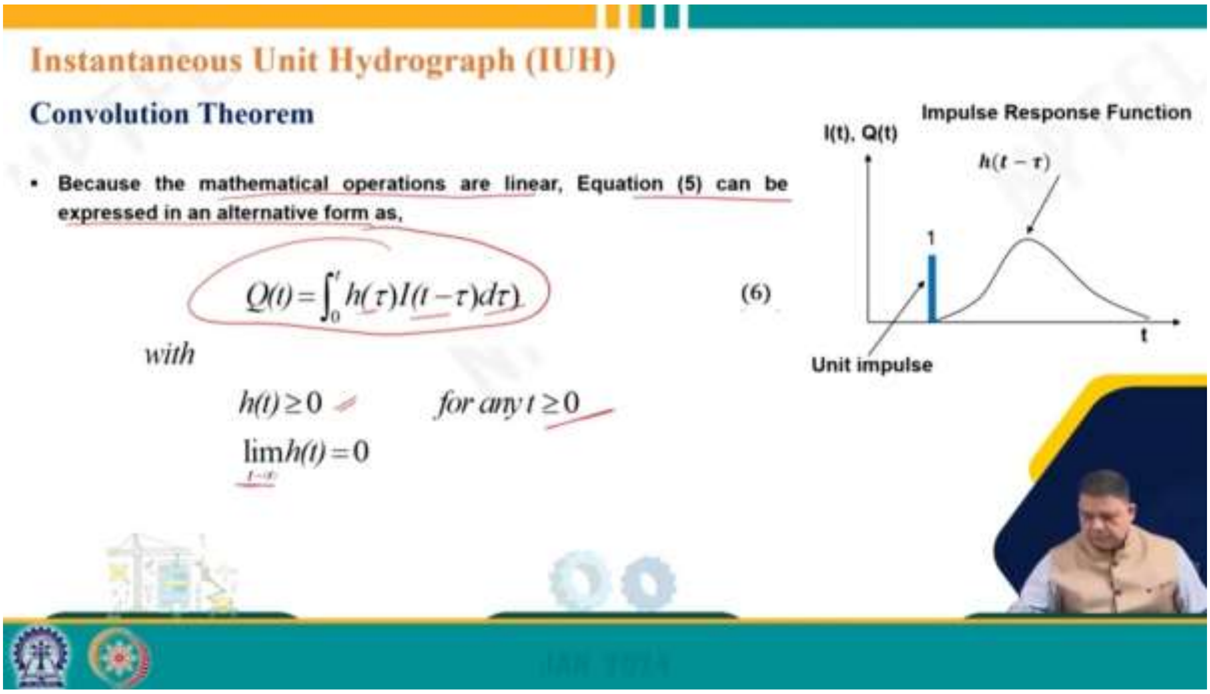


There is a convolution because there are two functions. So, that's why it is a convolution.

So, if $I(\tau)$ is intensity and $d\tau$ is infinitesimally small time, then $I(\tau)\,d\tau$ is the input during the interval. So, that is the magnitude of the rainfall that we are talking about. Thus, the resulting direct runoff will be $I(\tau)\,\mathrm{h}(t-\tau)\,d\tau$. So, this is the ordinate and this is the magnitude. We multiply $I(\tau)$ by the unit hydrograph, that is, the unit impulse, to get the magnitude. The response of the complete unit time function $I(\tau)$ can be found by integration.

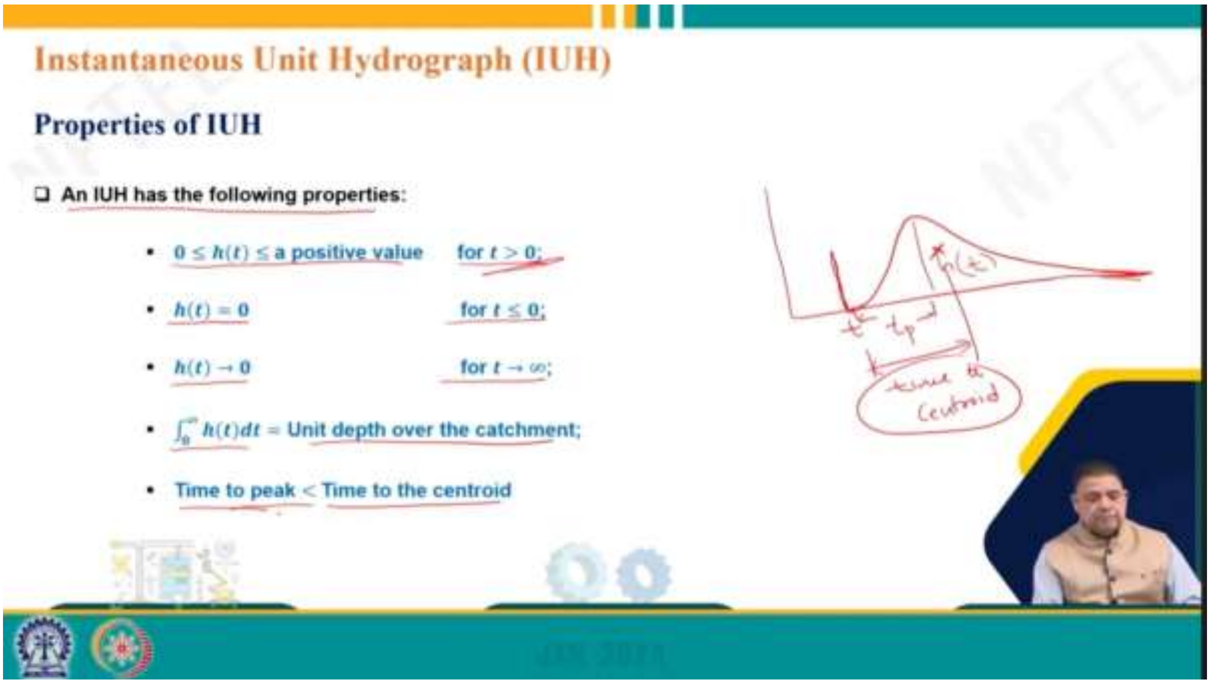
So, by integrating the resulting direct runoff, the following expression has been found. Here, $Q(t)=\int_0^t \mathrm{I}(\tau)\,\mathrm{h}(\mathrm{t}-\tau)\,\mathrm{d}\tau$. And $\mathrm{h}(\tau)=0$ for $\tau \leq 0$. So, that means, this $\mathrm{h}(t)$, $H(\tau)=0$. That is, the ordinate of this hydrograph will be 0 for $\tau \leq 0$. So, before that, it starts from there only.

So, that's what it means. Basically, this is called a convolution integral. This integral is referred to as a convolutional integral, which expresses the amount of overlap of one function as it is shifted over another function. So, there are two functions we are talking about unit impulse function and impulse response function. So, they are being blended here using this convolution integral, and that's why it's called that. IUH is based on the convolution theorem.



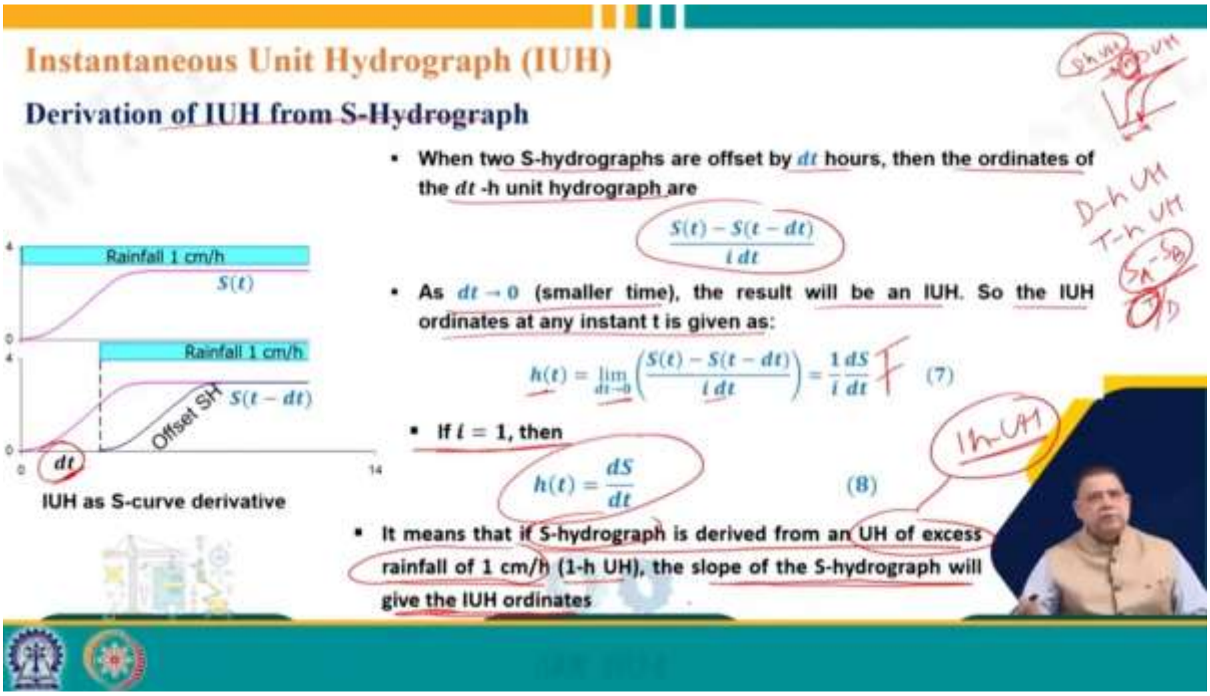
And because the mathematical operations are linear, equation 5 can be expressed in the alternative form. So, earlier, we were writing $I(\tau)$ h$(t{-}\tau)$, and now, mathematically, because it's the same, we are writing h(τ) $I(t{-}\tau)$ $d\tau$. So, that is the area under the curve, and with the condition that $H(\tau)$≥0 for any time t≥0. So, it will have some positive value, and $\lim_{t\to\infty} h(t) = 0$. So, as time progresses towards infinity, this approaches 0 value, and it is 0 before $\tau\tau$ for τ<0, as we already saw.

That's right. So, that means, this ordinate will have a positive value with time tt for any time greater than or equal to 0, but will approach 0 as the time approaches infinity. So, that's why it's a hydrograph basically, and that's why it starts from 0 and ends at 0.



Then we come to the properties of the unit hydrograph, and the unit hydrograph has the following properties: $0 \leq h(t) \leq$ some positive value for $t>0$. So, it lies between 0 and a positive value for $t>0$, which is quite obvious. $h(t)=0$ for $t \leq 0$. So, at t, at which... that's what we saw, that if this is the impulse here, from here only it will have... So, the first one says that $h(t)$ lies between 0 and a positive value, some positive value for $t>0$. It is t here in this case. $h(t)=0$ for $t \leq 0$, from 0. This is your $h(t)$ we are talking about. So, it will be 0 before this t, and $h(t) \rightarrow 0$ for $t \rightarrow \infty$. So, that is what we are talking about, the right tail approaching 0, and the area under the curve, that is, unit depth over the catchment, is the same way as the unit hydrograph we are talking about. And time to peak is less than the time to centroid.

So, that means, the centroid will be somewhere here, time to peak is less. There is a time to peak if you say, so time to peak is less than the time to centroid if you call it time to centroid. And again, this reminds us that the unit hydrograph or hydrograph is a skewed distribution, skewed to the right, with the right tail becoming long, remaining longer than the left tail. So, that's what it is ensuring here in this case, in this property.

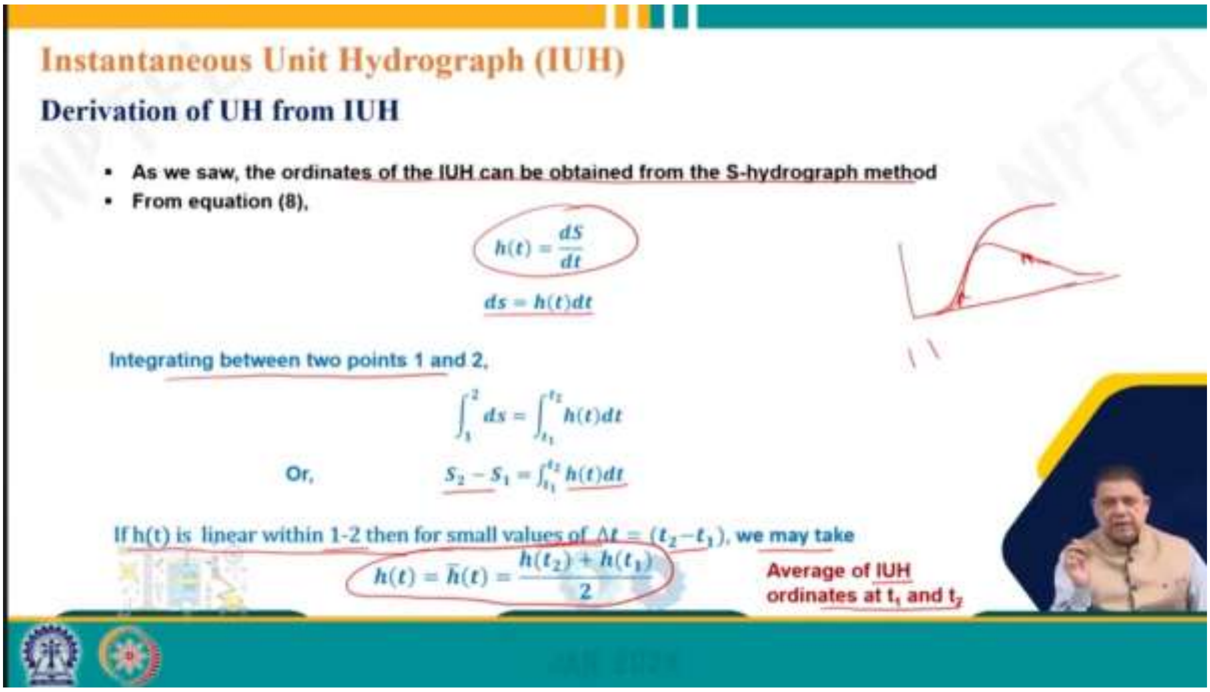


Coming to the derivation of the instantaneous unit hydrograph (IUH), IUH can be derived by using several methods. Some of the methods are listed here. Like, it can be derived using the S curve summation hydrograph, which we discussed earlier, conceptual model. There are some conceptual models of the unit hydrograph, instantaneous hydrograph, that is, Clark's method and Nash's models. Then we can fit the harmonic series to DRH and ERH for deriving IUH, and theoretically using Laplace transform function.

So, there are many various ways we can derive IUH, but we will be concentrating on the first one, that is, the S hydrograph method of deriving the unit hydrograph. And let us discuss the derivation of IUH from the S hydrograph. S hydrograph, we have already discussed, when two S hydrographs are offset by dt hours, then the ordinates of the dt hour unit hydrograph are given by this relationship. So, here $\{S(t)-S(t-DT)\}/idt$, that is the difference between these two that is what we discussed here, we are saying. Here we are saying that a one-hour unit hydrograph is one hour, one centimeter, but it is true for any duration. But S hydrograph can be derived by utilizing any duration unit hydrograph, which we have also seen earlier in examples. dt→ 0, the result will be an IUH.

So, the IUH ordinates at any time T are given by this equation: h(t) = $\lim_{dt\to 0}\left(\frac{s(t)-s(t-dt)}{idt}\right)$, or simply, h(t)= $1ds/idt$, where dt is a very small time, approaching 0. So, if we approach 0 this is how it can be.

It is nothing but S_A-S_B/(T/D) can be 1ds/idt, and if I = 1, then h(t)=dS/dt. So, it means that if the S hydrograph is derived from an IUH of Texas rainfall of 1 centimeter per hour, that is, 1-hour UH, the slope of the S hydrograph will give us the IUH ordinates. So, if we derive, we take a 1-hour unit hydrograph, develop the S hydrograph, and then if we find out the slope of the S hydrograph, it will give us the ordinates of IUH. So, that is a concept that has come to, I mean, such a high level, I could say because we already discussed all these processes. The only condition we are saying is that a UH of excess rainfall of 1 centimeter. That means we have to use a 1-hour UH, that is the only condition.

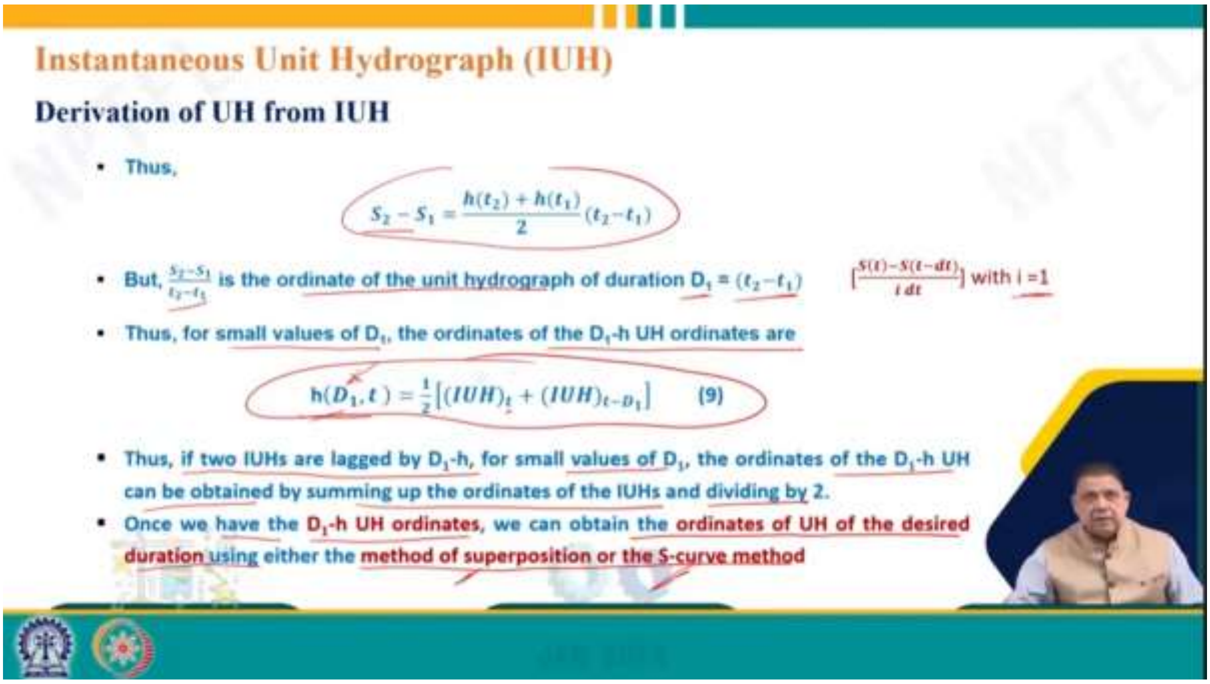


So, if we use the S curve, a 1-hour UH to develop the S curve, then the slope of the S curve will give us the IUH ordinates. So, it is a pretty simple way of developing the instantaneous unit hydrograph, and we saw that the ordinates of IUH can be obtained from the S hydrograph method. So, h(t)=ds/dt , where ds = h(t)dt, integrating between any two points 1 and 2. That means, integration on both sides,$\int_1^2 ds = \int_{t1}^{t2} h(t)dt$.

S_2-S_1=$\int_{t1}^{t2} h(t)dt$ So, if we say that any short duration if we take for hydrograph, that is nothing but we are talking of a hydrograph.

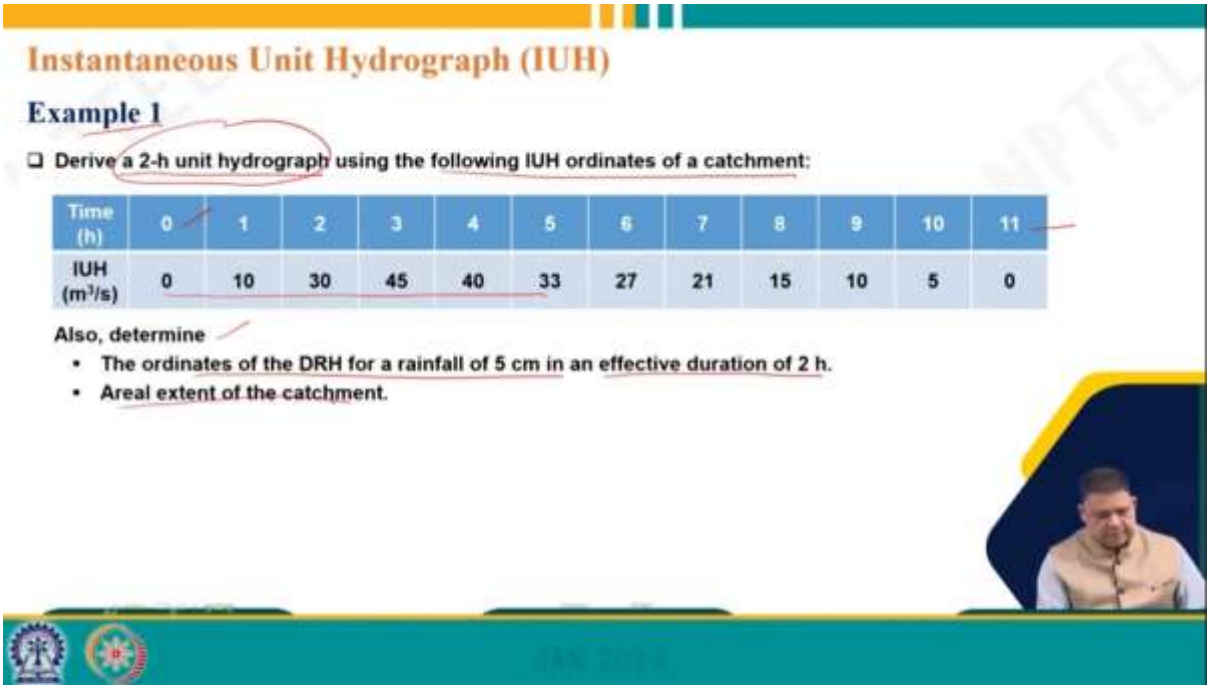
So, here, or any hydrograph we can say, or S curve we can say, ultimately because from S curve we will get a ΔS, we will give us a hydrograph. So, basically, if h(t) is linear within the. So, if we talk about very small-time intervals, then we can say that the curve, the curve will be here, can be taken as a straight line. So, if h(t) is linear within 1 to 2, then for small values of $\Delta t =$

(t_2-t_1), we may take that $h(t)= \overline{h}(t)= \{h(t_2) + h(t_1)\}/2$, or the average of the IUH ordinates at 1 and 2. So, 2 times intervals, if they are very small, then we know the ordinates are there 2 times. If we take the average, then that will give us the average of these IUH ordinates.



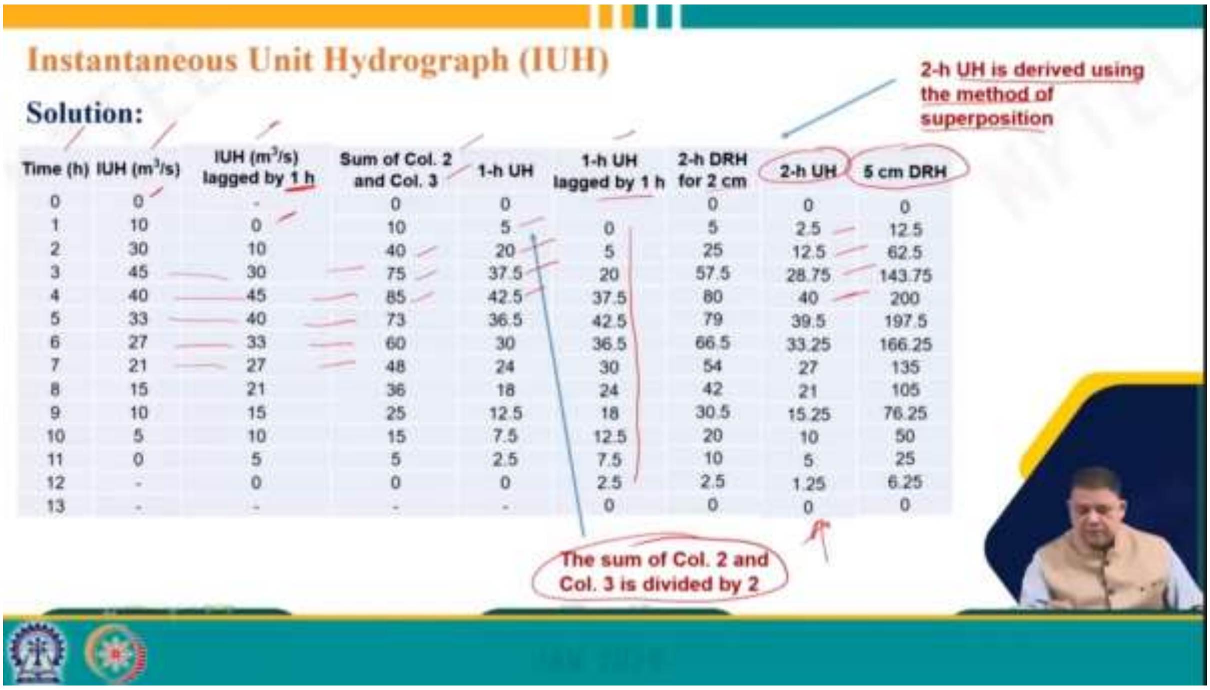
So, S_2-S_1= $\{h(t_2)+h(t_1)\}*(t_2-t)/2$ is this, already we know $(S_2-S_1)/(t_2-t_1)$ is the ordinate of the unit hydrograph of duration $D1= (t_2-t_1)$, that is what we discussed with i=1. So, for small values of $D1$, the ordinates of $D1$ unit hydrograph $h(D_1,t)$ can be obtained from this relationship: $½[IUH_t-IUH_{t-D1}]$. That is, if you get the IUH ordinates at t and t-D_1, where D_1 is very small, then it will be nothing but the average of 2 IUH ordinates will give us the ordinates of unit hydrograph. So, if 2 IUHs are lagged by D_1 hour for a small value of D_1, the ordinates of the D_1 unit hydrograph can be obtained by summing up the ordinates of IUH and dividing by 2.

Once we have the D-hour UH ordinate, we can obtain the ordinates of UH of derived duration using either the method of superposition or the S curve. So, if we have an IUH, then we can develop a small duration $h(D_1,t)$ unit hydrograph by using this concept. Once we have a unit hydrograph of any duration, then we can, for any other duration, we can develop either using the method of superposition or using the S curve, which we have discussed earlier. I think mathematically it might look a little bit complex, but we can easily take an example and discuss this.



Time (h)	0	1	2	3	4	5	6	7	8	9	10	11
IUH (m^3/s)	0	10	30	45	40	33	27	21	15	10	5	0

So, let us take an example here, derive a 2-hour unit hydrograph using the following IUH ordinates for the catchment. So, IUH ordinates are given between 0 and 11 hours, and as you can see here also, determine. So, we have to determine a 2-hour



Time (h)	IUH (m^3/s)	IUH (m^3/s) lagged by 1 h	Sum of Col. 2 and Col. 3	1-h UH	1-h UH lagged by 1 h	2-h DRH for 2 cm	2-h UH	5 cm DRH
0	0	-	0	0		0	0	0
1	10	0	10	5	0	5	2.5	12.5
2	30	10	40	20	5	25	12.5	62.5
3	45	30	75	37.5	20	57.5	28.75	143.75
4	40	45	85	42.5	37.5	80	40	200
5	33	40	73	36.5	42.5	79	39.5	197.5
6	27	33	60	30	36.5	66.5	33.25	166.25
7	21	27	48	24	30	54	27	135
8	15	21	36	18	24	42	21	105
9	10	15	25	12.5	18	30.5	15.25	76.25
10	5	10	15	7.5	12.5	20	10	50
11	0	5	5	2.5	7.5	10	5	25
12	-	0	0	0	2.5	2.5	1.25	6.25
13	-	-	-	-	0	0	0	0

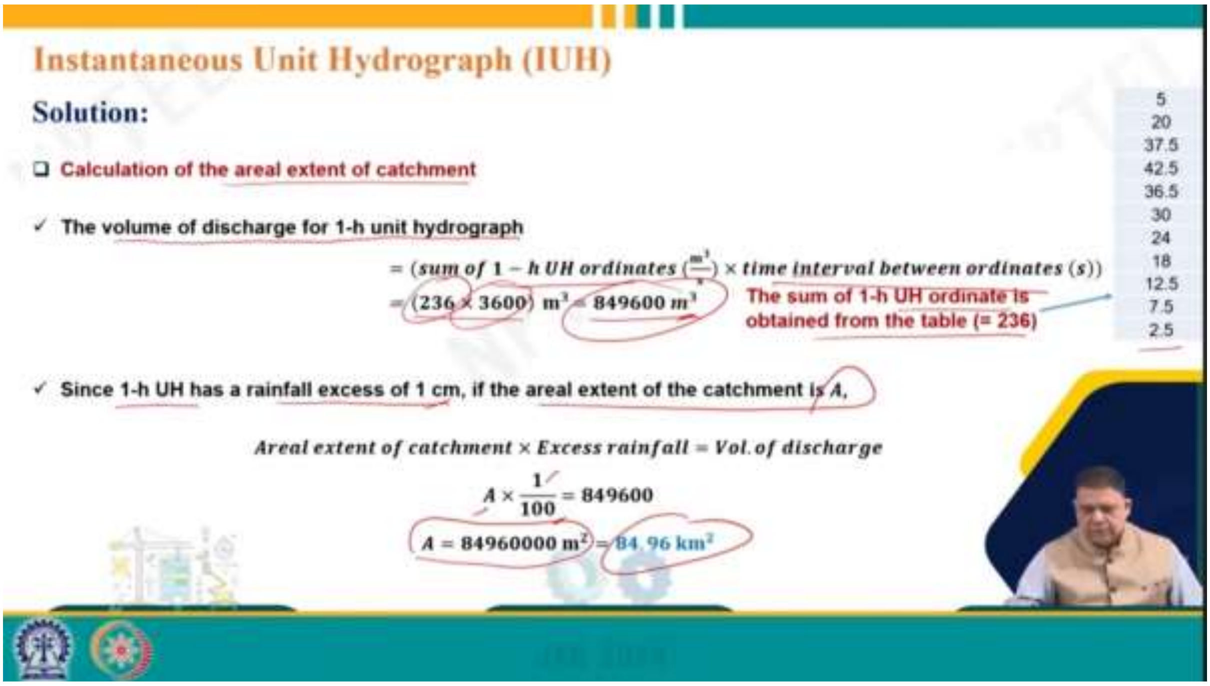
So, we have our time, and the IUH ordinates are given here. To develop the 1-hour UH, what we will do is lag this IUH by 1 hour. So, here we start from 0; here it will start from 1 hour, and the entire thing will be reproduced. The sum of these two columns, columns 2 and 3, will be found: 0, 10, 40, 75, 85, 73, 60, 48, and so on. Then, the sum of column 2 and column 3 will be divided by 2. That means, we will take the average of these two IUH ordinates, which are lagging by only 1 hour, a small value we are considering. Then we will get the 1-hour UH ordinates.

So, 10 divided by 2 is 5, 40 divided by 20, 75 divided by 37.5. So, from the given IUH, we have developed a 1-hour unit hydrograph, and we were asked to develop a 2-hour unit hydrograph. Once I have a 1-hour unit hydrograph, I can develop the 2-hour unit hydrograph by the method of superposition, lagging this 1-hour hydrograph by 1 hour, and then superimposing them.

So, the 1-hour unit hydrograph lags by 1 hour. That is what we have developed; we are lagging by 1 hour here, and the 2-hour unit hydrograph for 2 centimeters, will be the sum of these 2. So, 0 + 5, 20 +5 is 25, 57.5, 80, 79, and so on. It is a 2-hour DRH, and then we must develop a 2-hour unit hydrograph. That means we need to divide these ordinates by 2.

So, 5 divided by 2 is 2.5, 12.5, 28.75. 80 divided by 2 is 40, and so on. This is how we have got this 2-hour unit hydrograph. So, this is the first thing we want to develop. Then we were asked what would be, if 2-hour, in 2 hours, 5-centimeter effective rainfall occurs, what will be the ordinates of DRH. This, of course, I have mentioned already, that 2-hour ordinate is derived using the superposition, meaning we lagged 1-hour unit hydrographs and then found out.

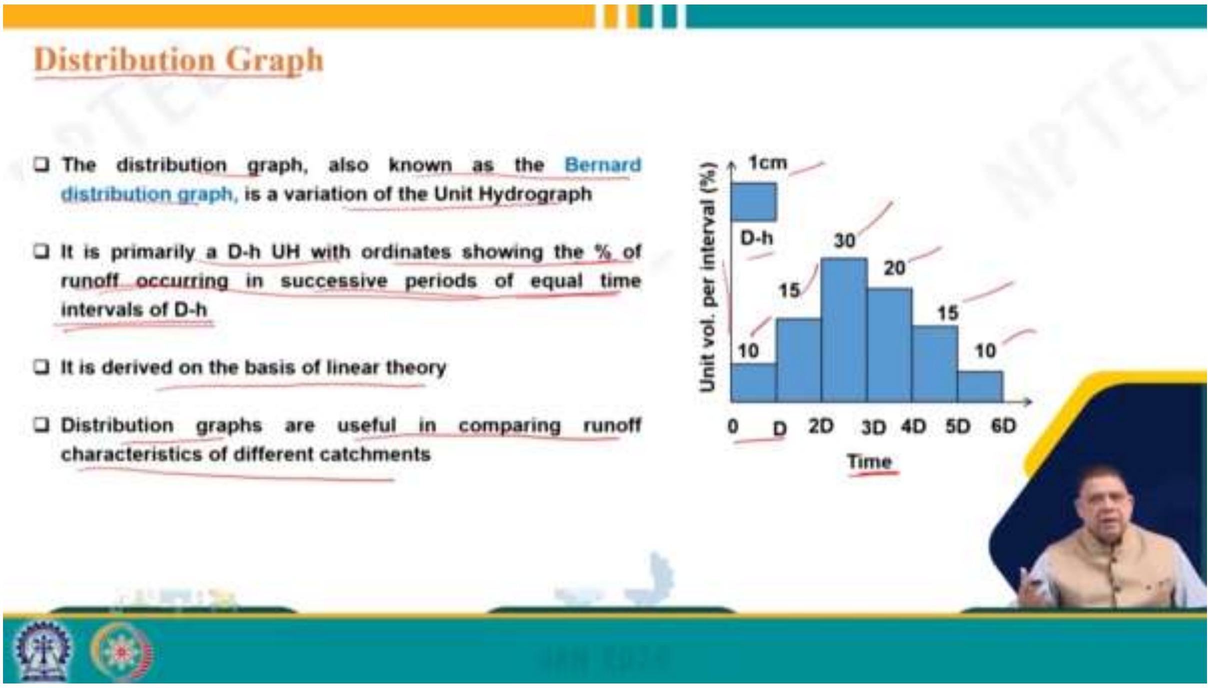
For 5-hour DRH, 5-centimeter DRH, we can, once we have the 2-hour unit hydrograph, 5-hour DERH, we have to simply multiply these ordinates by 5. So, 12.5, 64, 2.5, and so on. This is how we can get what will be the magnitude of DERH if effective rainfall is 5 centimeters.



Then we must find out the aerial extent of the catchment. For that, we have to use the concept that the volume of discharge for a 1-hour unit hydrograph represents 1 centimeter. So, the volume can be found out by the triangular area because we have a unit hydrograph of triangular and tapering shape. So, we know that relationship. The area comes out, the volume of discharge comes out to be 849,600 cubic meters. The sum of 1-hour ordinates is 236 from here. So, that is why we are using 236, and then the time interval is 3600 in seconds.

Since the 1-hour UH has a rainfall excess of 1 centimeter, if the aerial extent of the catchment is A, then if you multiply 1 centimeter with A, we will get the same volume of discharge that we calculated from the area under the curve. $A \times 1/100$ - this is in because this is in square meters, So, that is why we have to convert it to meters, and that is equal to this. So, the area we get is this many square meters, or we can say that the area is 84.96 square kilometers. So, that is what we will get in this case. This is how we can find out the area also. So, from your 1-hour IUH, we first developed the 1-hour UH, and then we developed the 2-hour UH using the method of superposition, and then we found out if effective rainfall is 5 centimeters, what will be the DHR units, and then also using the concept that the unit area under the unit area graph represents 1 centimeter of effective rainfall. So, using that concept, we also found out what is the area of the catchment.

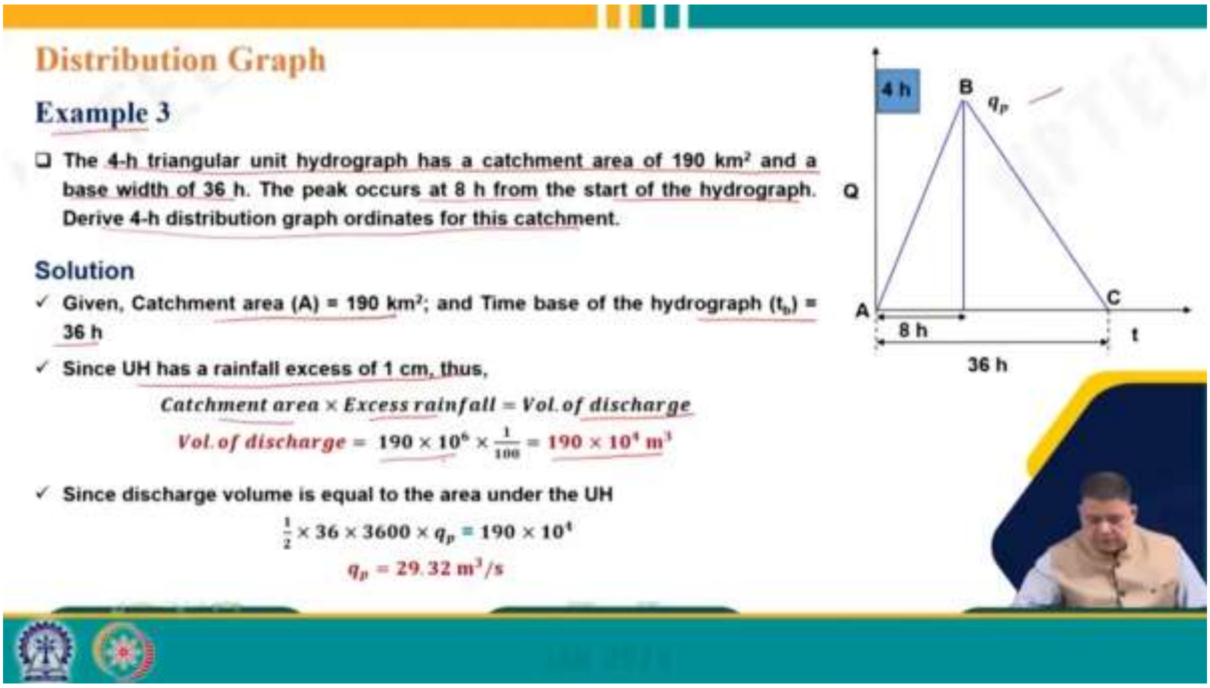
This is how we can handle a strategy in hydrography.



Now, we come to the second part of the lecture where we will talk about the distribution graph. The distribution graph, which is also known as the Bernard distribution graph, is a variation of the unit hydrograph, meaning it is a variant of the unit hydrograph. It is primarily a DR unit hydrograph with ordinates showing the percent of runoff occurring in successive periods of equal time intervals of D hours. So, earlier, we were plotting a graph, so here the ordinates is a volume per percent volume, which represents time instead of discharge.

So, here 0 to 10, 0 to D hours, D hour 1 centimeter. So, in 0 to D hour, 10 percent of the runoff occurs; in the next to 15 percent, in the next D hour, 30 percent, and so on. So, the sum will have to be 100. So, it is here 20, 15, 10, and so on. This is the percent of runoff occurring in successive periods of equal time intervals of D hours, that is what is represented by this distribution graph. It is derived based on linear theory, meaning we consider that it will occur uniformly.

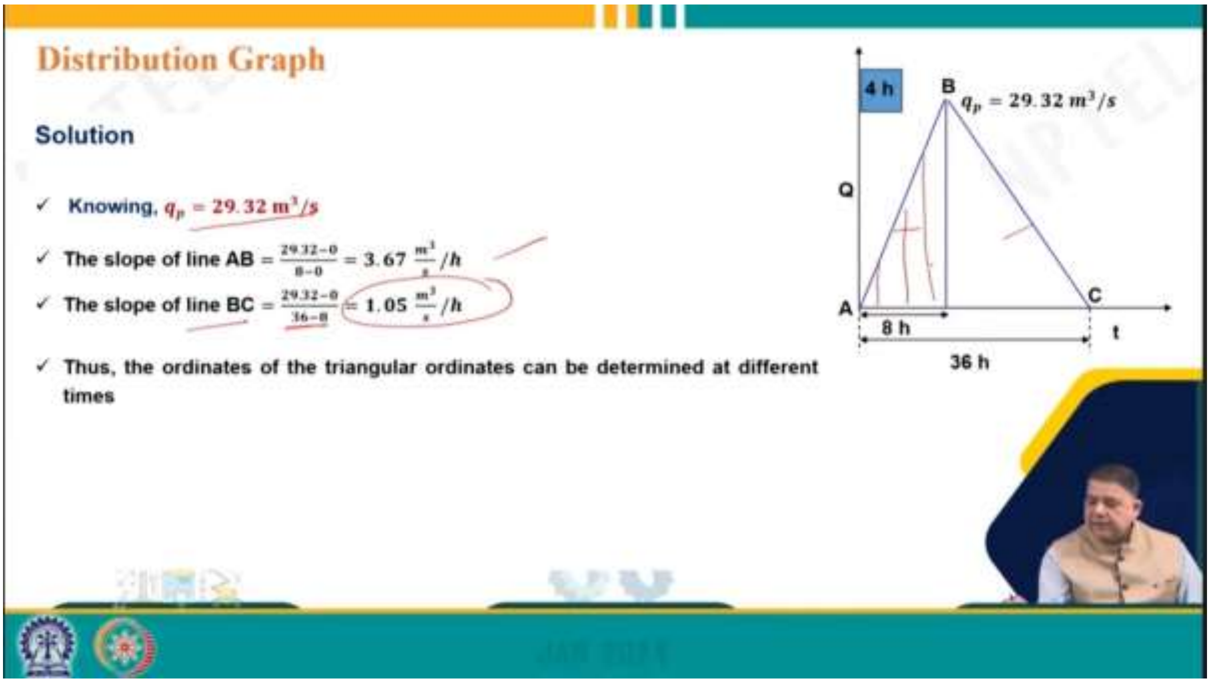
Distribution graphs are useful in comparing runoff characteristics of different catchments. So, if we know the catchment characteristics and we have distribution graphs from different watersheds or catchments, then we can find out how a similar magnitude of rainfall, similar duration, and how the rain runoff is distributed over time, can be seen in the case of a distribution graph.



Let us take an example, example 3, the 4-hour triangular unit hydrograph has a catchment area of 190 square kilometers and a base width of 36 hours. The peak occurs at 8 hours from the start of the hydrograph. Derive a 4-hour distribution graph ordinates for this catchment.

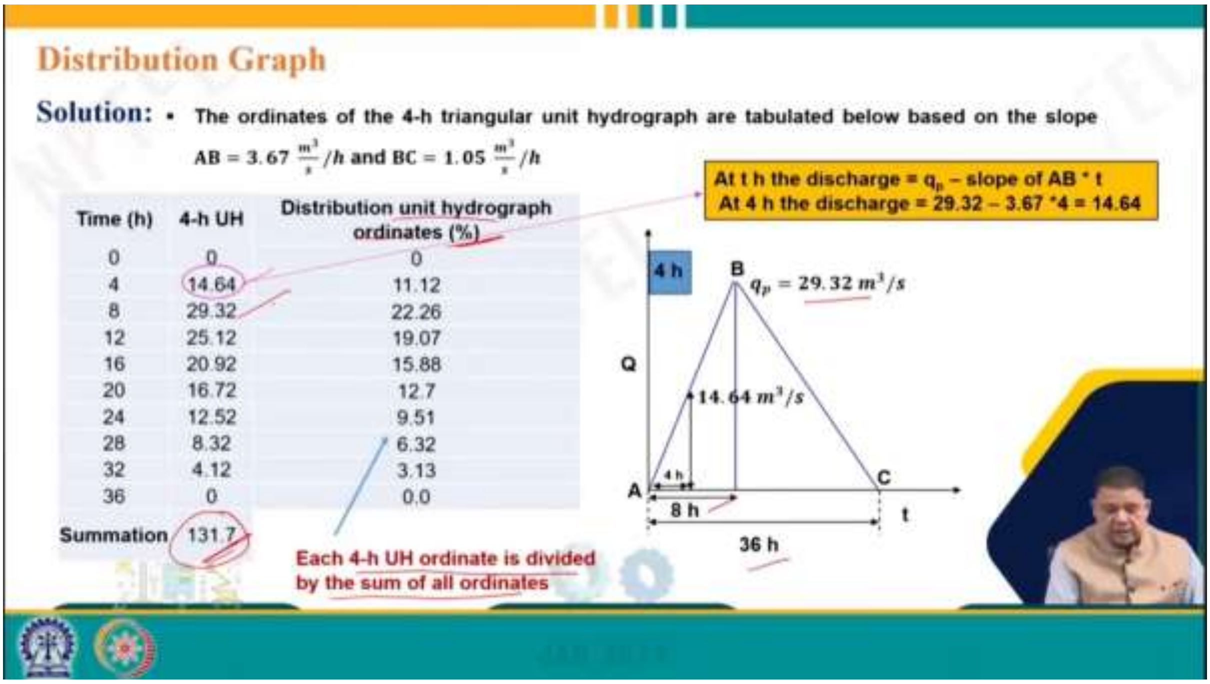
So, this is what is given, the graph is plotted. We have been given a catchment area of 190 square kilometers, a time base of hydrograph at 36 hours. Since the UH has a rainfall excess of 1 centimeter, so we use the same concept, catchment area into excess rainfall is volume of discharge. And the volume of discharge we can find out the area here because q_p is 190.

So, half into 190 is an area into 1 unit depth. So, this is the volume of discharge, and the volume of discharge is also equal to the area under the curve that is shown here. So, these two are equal. So, using this, we find out the value of q_p, which comes up with 29.32 cubic meters per second.



Now, knowing q_p and this timing, we can find out the slope of the lines AB and BC. So, AB is 29.32- 0/8, which is 3.673.67 cubic meters per second per hour, and the slope of line BC is 29.32-0 over the denominator, which is 36 hours- 8 hours, which is 28 hours.

So, that comes out to be 1.05 cubic meters. Now, because we know the slope, at any hour we can find out the ordinates of this 4-hour unit hydrograph.



So, here are the values: 29.32, the 8-hour, 36-hour, and the different hours we know. That means, basically we are using the slope of these 2 lines for getting the sum, which comes out to be 131.7. Now, the distribution unit at hydrograph ordinate will be percent. That means each 4-hour UH ordinate is divided by the sum of all the ordinates.

So, that means, 14.64 divided by 131.7, 29.32 divided by 131.7, and so on. So, every ordinate will be divided, and then we plot the distribution graph, which is the distribution of hydrograph ordinate percent at different 4-hour durations. So, for the first 4 hours, it is 11.12, for the second 4 hours, 22, and obviously, the sum will be 100. So, this is the distribution hydrograph. Today, in this lecture, we discussed the stationization hydrograph and distribution hydrograph, and we saw the practical applications.


THANK YOU