Evolutionary Dynamics
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Hi, welcome everyone to this next video. So in the last video, we saw that we can model bacterial growth in a batch culture or in any environment using an exponential growth curve. But very soon, we'll find out, as we found out through the calculations done towards the end of the lecture, that very soon we'll run into some problems. And the problem is that exponential growth facilitates growth so fast, and the numbers increase so rapidly, that whatever environment in which this growth is taking place will very soon prove to be insufficient to support the growth of that large a population. As we saw through that toy calculation, in a time as small as three days, the bacterial population, if assumed to be exponential, increases to such an extent that its combined mass exceeds that of the mass of the planet.

So clearly, the exponential phase, although very important, is only applicable for a short duration in the growth phase of a bacterium. So what we'll do in this lecture is discuss two other models associated with growth, which allow us, to some extent, to make more realistic assumptions about how growth is going to take place. So here in the slide is what we discussed in the previous video: exponential growth, and this was the problem—that this shoots up to an unrealistic number. And that's a problem.

What we are going to do today is a model called logistic growth. And what this model tries to do is it tries to learn from the physics of what is happening, the physiology of what is happening, and tries to incorporate that in a mathematical form. To understand what is going on, let's once again revisit the typical growth curve of bacteria in a batch culture. Let us assume that we start with N naught number of bacteria, and we saw that in this batch culture, this is what is going to happen.

And let's not worry about what happens beyond this because really we're only interested in the period of time where growth is supported and the culture reaches stationary phase. Of course, many people study what happens in and after stationary phase, but let's stick our concentration to only lag, log and stationary phase, how a bacterial population arrives at the stationary phase. So this is what we know from experiments that typical growth curve is going to look like. Now let's try to

imagine this in conjunction with what is happening in this flask. So here we have our flask, it's shaking, there are nutrients available and we seed this with a bacteria.

And this bacteria is going to divide which is going to divide again and so on and so forth. And this exponential phase is depicted by this exponential increase in number of bacteria in the flask. However, we note that there is a deviation from this curve as time moves forward. If this curve was truly exponential, then the number of bacteria would have moved along this trajectory.

If it was truly exponential from N naught, this would have been the curve. However, what's happening here is that there is a slowdown in growth when we compare the actual growth of how numbers increase to a purely exponential growth. So this is purely exponential and this is actual growth. So, if we compare these two, we note that exponential and actual growth dynamics only match each other for a small window of time from 0 to T naught. And beyond T naught, the actual growth slows down and very soon it completely halts.

And right here, growth is stagnant, so there is no growth in this phase because the numbers are stationary. So the growth rate beyond a certain point deviates from exponential, slows down, and keeps slowing down until it reaches zero. So let's record these observations and see how we can mathematically understand them. So in the initial phase, What this tells us is that it's perfectly fine to assume that growth is exponential.

However, this initial phase, let's say, goes from 0 to t naught. However, as we move beyond t naught, Growth is no longer exponential. It is, in fact, slower than exponential. To an extent that after a certain point, it slows down to a point where growth actually halts to zero.

Growth completely halts. And eventually, it reaches zero. And this slowdown—the extent of slowing down—keeps increasing. So the extent of slowing down keeps increasing. As we move from time T naught to time T1, at which point growth is virtually undetectable in this growth media that I'm testing this model in.

And remember, all of this is for a batch culture. All right, so these three characteristics is what we have to satisfy when we have to talk about a model which represents that exponential phase is not going to last for very long, but in fact, growth, actual growth will very soon be slower than exponential growth. So how do we tackle this? Mathematically, this, Incorporating these three aspects that we just discussed are done by only including one term in the model that we already discussed.

So, this is dn by dt equals Rn. We already discussed this. This, in fact, leads to exponential growth. but we have an additional term here, which is one minus N upon K. By just addition of this

additional term in this model that we already studied, we can incorporate all three conditions that we discussed in the previous slide, that in the initial phase growth is roughly exponential, Beyond a certain point, it slows down and becomes slower than exponential.

And eventually it slows down to an extent that growth reaches zero. All three of these characteristics are included in the mathematical form of bacterial growth. If we just multiply our exponential model with this term one minus N by k. So what does this term do and what does this term represent is what we look at now. So to understand that, let us first discuss what is this variable K. K is called the carrying capacity of the environment.

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

Of the environment. If you remember a few videos ago, we did this example of thinking of E. coli growth in a flask whose volume was 100 milliliters. It contained growth media, which is called LB, and we saw that after a sufficiently long time of growth in LB, the number of bacteria in this flask reached 10 to the power of 11. That is because in an LB culture, 1 milliliter of the liquid media can support growth of up to 10 to the power of 9 bacteria. So, 1 milliliter supports growth of 10 to the power of 9.

After 10 to the power of 9 has been reached for every milliliter of culture, all the nutrients that are present in the flask are gone, and there are no more nutrients left to support growth. So, this is the limit of the environment. To the extent that it can support growth in this flask per milliliter. So, if you have 100 milliliters, then the total number of bacteria that can be reached is 10 to the power of 11 and not beyond that because the environment simply doesn't have sufficient nutrients to support more growth. This number is called the carrying capacity of an environment.

That is how many individuals an environment can support, and beyond that, it cannot go because it simply runs out of nutrients that it contains. So, how many individuals an environment can support is the carrying capacity of the environment. So, this automatically takes into account the the quality of the media and the quantity of the media. For instance, if we have LB, 1 ml at 37°C with shaking, then K is 10 to the power of 9.

However, if we have 100 mL at 37°C with shaking, then the K associated with it is 10 to the power of 11. If this wasn't LB, if the carbon that I supplied was in the form of glycerol, 100 mL, with all these conditions remaining the same, then the carrying capacity of the environment will be 10 to the power of 10 because it won't support growth to the extent that LB can. And so on and so forth. So K, the carrying capacity of the environment, represents two aspects of it.

The carrying capacity of an environment represents the quality of the media. For instance, in this case, 100 mL of LB supports more organisms as compared to 100 mL of glycerol. So this is reflective of the quality of the environment that we are supplying. It is also a function of the quantity of the resources present in the environment.

For instance, if we have 100 mL of the same resource versus one mL of the same resource, the K associated with them is also 100-fold. So you multiply the resources by 100, and we get 100 times the carrying capacity of what it was when we only had one mL of LB. So carrying capacity is a function of quality and quantity. It's also a measure of quality. Quality is not reflected by only the chemical constituents of the environment.

Chemical constituents. For example, it's not just the difference in hair, it's LB and it's glycerol hair, but also the physical environment, the physical conditions of the environment. For example, if I had 100 mL of LB, but at 75 degrees Celsius, then the carrying capacity of that environment would be zero because E. coli wouldn't be able to reproduce in those environmental conditions. So it's important to note that carrying capacity represents the carrying capacity of an environment as a function of the quality and quantity of that environment. So that's the K here.

This N and this N represent the same thing, which is the number of individuals at time t. So this keeps changing with time as the number increases. So let's now see how this equation captures the three conditions that we listed on the previous slide, which were again these three. Initial phase growth is exponential. Beyond t_o, growth is slower, and growth halts and eventually reaches zero as numbers keep on increasing. So we'll spend some time looking at how this equation helps us achieve that.

Okay, let's write the equation again. dN by dt equals RN. One minus N by K. And let's assume that the environment we are talking about is the following. It's LB, 100 milliliters. Growth is at 37 degrees Celsius.

There is shaking. That's the condition, the initial conditions. N naught, the number of bacteria at T equal to zero, is just one. And what this automatically implies is that K is 10 to the power of 11 because every ml of LB can support 10 to the power of 9 bacteria. And let's, for the sake of simplicity, just say R is equal to 1.

So, given these conditions, let's see how this plays out. So, when I start the growth experiment, I'm starting with One bacterium, this grows, divides into two, and so on and so forth. And then there is this, what is called, this branching process, and the numbers keep increasing exponentially until

they reach the carrying capacity. So, starting with one, I reach K. And that's how the numbers in the population move as time increases.

And this is the number of bacteria in the flask. And time moves in this direction. So, when we are first starting out, let's divide this into three phases and try to see what happens. In the first phase, this is close to the start of the experiment. This equation would be dN by dt equals Rn times 1 minus N by K. However, we know that close to the start of the experiment means

It is important that we are approximating this equation, but under these conditions close to the start of the experiment. We know that K is a constant for these given conditions, which is 10 to the power of 11 and close to the start of the experiment. We started off with one, then it will become two, and then it will become four, and so on and so forth. So, N is very small. Let us say it is of the order of 10.

So, even if n is 1, although we start with 1, 10 is still 3 or 4 generations away. This equation becomes Rn times 1 minus a very small number, which is of the order of 10, divided by 10 to the power of 11. This is roughly equal to Rn times 1. 1 minus 10 to the power—this 1 gets canceled—this is 10 to the power of 10, and then we bring it to the numerator, this is 10 to the power of minus 10. This is roughly equal to Rn times 1 minus 0.00000001. which is 0.99999999 times Rn.

And if you see, this is very, very close to just Rn. The difference between these two is so small, much less than 1%. In fact, this is like one-thousandth or one ten-thousandth of a percentage, even smaller than that. So, the difference between these two is really, really small. Hence, close to the start of the experiment, growth can be approximated. If we are using this particular formula, then growth can simply be approximated as Rn.

This approximation is only valid if this is close to start of experiment and our starting numbers were really small. If our starting numbers were really large, then we wouldn't be able to neglect this particular factor. So what this equation predicts is that at the start of experiment, at start of experiment, If N naught is much smaller than K, then growth can be approximated as exponential. If we go back a couple of slides and see this is what we had written here that in the initial phases growth is exponential is what I see from experiments that I do.

So this equation successfully captures that part of it that growth in the early phase is close to exponential. Let's go to the next phase. Now we will see when population is close to but still less than carrying capacity. Now, let us imagine a scenario where N is roughly K by 2.

So, if so, we saw that k was 10 to the power 11 that means N is approximately half into 10 power 11 which is just 5 into 10 power 10. In such a scenario, what you should note is that this fraction

becomes roughly if n is approximately k by 2, then this fraction becomes this fraction is approximately half because we get 1 minus n upon k as simply equal to 1 minus, instead of n, I will replace K by 2, 1 minus K by 2 divided by K, which is just, the K's cancel, 1 minus half, which is just half. What that means is dN by dt is Is equal to.

R upon 2. N. So this predicts. That when. The number of bacteria is. Half the carrying capacity.

At that point. Population growth is still exponential. But. The growth rate. has reduced from R to R by 2 and so on and so forth.

And we can see that if N was K by 3, then growth rate R effective would be not R, but 2R by 3 and so on and so forth. So as this N is increasing, the R effective is decreasing. In this case, R effective is just R by 2. This satisfies the second criteria that we were looking at, which is that beyond t naught, growth is slower than exponential as defined by this particular exponent. Growth is getting slower as numbers are increasing.

That defines the second criterion. And the third one says that as n approaches k, In this population, as n approaches k, growth should completely halt. And that is easy to see through this model because dn by dt in that case would be—I mean, the equation we have is dn by dt equals Rn minus 1. Rn times 1 minus n by k, but in the scenario where n approaches k, so n is very, very close to k. In that scenario, if we look at dn by dt,

That is just equal to R n times 1 minus a number which is almost k divided by k. This fraction here is 1 minus effectively 1. Hence, it comes to 0. So, when n is equal to k, dn by dt. is equal to R n times 0, which is 0, which means if dn by dt is equal to 0, this means that as t changes or as t moves forward, the rate of change of n is 0. As t moves forward, n does not change with time because the rate of change of n is 0, which means that n remains constant with time.

This is only true once it reaches k. So we have seen that by looking at this logistic equation, we can see that it satisfies what we actually observe in growth curves when I perform experiments. That initially, growth is almost exponential. As numbers increase, growth slows down, and the exponent reduces. And finally, as the batch culture that I'm talking about reaches saturation, growth completely halts and approaches zero.

So just by incorporating this particular variation to the exponential equation that we started with, we can see that growth is now able to capture all these facets that I see experimentally happen. In this context, we can do one more thing. Imagine that I have species A, and another species called species B. And I co-culture them together in the same flask. So this is a flask which has been seeded with one individual of type A and one individual of type B.

And the challenge before us is that as time moves forward, how are we able to predict what happens to the number of A individuals and what happens to the number of B individuals as time moves forward and this flask reaches saturation? A minor modification in the equation that we just studied will allow us to do that. So in that case, we will write an equation that dNA by dt equals So, we will write two equations here. The first one is for the first species: dNA by dt is simply equal to RA NA times 1 minus NA plus NB divided by K. Let me first write down both equations and then I will explain what these actually mean.

Similarly, for the other species, dNB by dt is equal to RB NB times 1 minus NA plus NB divided by K. So, what do these terms mean? The K here is the carrying capacity of the flask in which growth is happening. RA is the growth rate of species A. If species A was present by itself, this is how fast it would grow. Similarly, RB is the growth rate of species B.

NA and NB are the number of individuals belonging to species A and species B. In this term, in the bracket here, we have to add both of them. Because remember that K is the total number of organisms that the environment can carry. Those K individuals could be of species A or species B; it doesn't really matter, but the total of them cannot exceed K. The fractional capacity of the environment that has been used up is equal to NA plus NB. This is the total number of individuals at any given point in time that the environment is already supporting.

The total number of organisms that the environment can support is K. So this fraction here represents what fraction of carrying capacity has already been used up. And obviously, when we subtract this particular fraction from one, it tells us how much fractional carrying capacity of the environment is still left in terms of resources being available to support further growth. If we solve these two equations together—and we won't get into that too much here—but if we solve these equations together, we can get the profiles of how NA changes with time and how NB changes with time.

These have to be solved together because, if you note here, this is an equation for NA, but it contains NB. And NB is also a function of time. It's changing with time. Similarly, this is an equation for NB, but it contains NA, which is also changing with time. So NB is dependent on NA, but NA is dependent on NB, and hence these have to be solved together to get the profiles of the growth of these two species together.

$$\frac{dN_A}{dt} = r_A N_A\left(1 - \frac{N_A + N_B}{K}\right)$$

And

$$\frac{dN_B}{dt} = r_B N_B\left(1 - \frac{N_A + N_B}{K}\right)$$

So in this video, we have talked about one more approach to quantify growth and look at the fitness of different strains when they are growing in a batch culture.