Evolutionary Dynamics

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Lecture 32

Hi everybody, welcome to the next video in the course. So in the last couple of videos, we discussed something called the marbles-in-a-jar game, which showed us how chance plays an important role in that game itself. But then we built an analogy of the game with how evolutionary processes occur, and we saw how each marble jar could be looked at as one generation. And subsequent jars show evolution taking place as we pass from one generation to another. And we saw that natural selection is not sufficient to explain evolutionary processes and that chance events also bring about evolutionary change. The smaller the population size, which was capital N, the greater the likelihood that chance will have a meaningful impact on the evolutionary dynamics of a population.

So small populations are at a greater risk of having their evolutionary trajectory dictated by chance as compared to natural selection. Large populations, on the other hand, are more dictated by natural selection as compared to chance events. Natural selection is able to act more effectively on larger populations. So what we'll do in the next two or three videos is formalize this structure and develop some mathematical relationships to understand these processes. As far as this entire course is concerned, the next three or four videos are going to be mathematically perhaps a bit challenging for those of you from a life sciences background.

But my honest suggestion to you would be to stick through these videos. We will take it very slow. We will try to develop the mathematics very slowly. And at the very least, even if the mathematical concepts prove inaccessible, we will discuss the geometry associated with the process that is going on. So even if you are not able to grasp the mathematical nitty-gritties, there are two things that I want you to take from these lectures.

One is the geometrical intuition of the processes that we are talking about. We should have some intuition about what the process is that is being talked about in a mathematical context. And it's okay if we don't get all of the mathematical nitty-gritties completely. The second is that whatever math we do at the end of the derivation process, we will derive some results. Those results have to be understood,

if not mathematically, then in terms of their significance toward evolutionary processes. So, we must realize what this result means. And these next results, the ones that we are going to derive in the next few videos, are exceedingly important for our complete understanding of evolutionary processes. So, with that background, let us start, and as I said, the next few videos are going to be in this spirit that I just spoke about. So, the broad question—we are going to be interested in two broad questions over the next few videos.

The first one that I am interested in is the following: if I have a chemostat and whose volume is V, the flow rates are V naught. And if this chemostat is populated by individuals of a particular type, then at a given moment in time, all individuals in this chemostat are of this black genotype. So, this is one particular genotype.

Let's call it B type individuals. And at this time, what happens is that a blue individual comes in the population and its fitness. This is the A type individual. which is carrying a mutation and this mutation may cause a change in fitness or may not cause a change in fitness. So, as far as fitness of this A type individual is concerned, we have three options that fitness of A may be greater than fitness of B, fitness of A is equal to fitness of B or fitness of A is less than fitness of B.

This, of course, is a case where the mutation that happened was beneficial in nature. This is a case. The second one is a case where the mutation that happened was neutral in nature. And the third one is a case where it was a deleterious mutation that happened. We will not discuss the deleterious case because in this case selection just takes care and eliminates this one mutant that has come up.

So this is not interesting because this A type individual is less fit as compared to everybody else in the population and selection is going to just remove it. So we will not discuss this case. We will discuss these two cases where this is a neutral mutation or this is a beneficial mutation. So So, this individual has arisen and this individual could be one of the two types, either neutral or adaptive.

The broad question we are interested in, as I said, is these two questions that we are going to answer mathematically. The first question is: What is the probability that this system transitions from what is shown on the left to the right? This state, so the question of interest is: What is the probability that, starting from only one individual of the blue type, the system reaches the state where all cells are of the blue type? What is this probability?

That is what we are interested in. In other words, this has a formal name: What is the fixation probability of the blue mutant? And we are going to answer this question in two contexts. One, when the blue mutant is neutral compared to the black genotype, and the other case where blue is adaptive compared to black. So I hope this question is clear to everyone, and that is what we are going to build on in the next few videos.

So before we do that, we want to understand the approach we will take to answer this question. We need a mathematical approach to answer this question. And again, you should realize that this probability of transitioning from this state to this state—this probability of transitioning—is not equal to one, even if blue is more fit compared to the black genotype. And that is because, as we discussed last time, since there is only one blue-type individual, there is a finite probability that this blue individual is washed away from the reactor even before it has had a chance to divide.

And once it's been washed away, then there are no more blue-type individuals left in the population. And hence, the only possible state for the system is all black-type individuals. And the probability of this transition is equal to zero. So one assumption that we are going to make throughout this discussion when we address this question is that no new mutations happen. We will discuss, once we are through with these derivations, how relaxing these assumptions will change our results—whether it will increase the probability that we calculate or decrease that probability.

But we'll discuss that at the end. But for now, we will assume that the blue and the black genotypes are the only ones that exist and no mutations happen. Blue gives rise to blue every time. Black gives rise to black every single time. And that's it.

So before we begin to develop this framework, what we'll do in this video is develop a mathematical process to analyze questions and systems such as this. And this technique is called the Moran process. In a Moran process, from the context of our chemostat, the state of the system is this, and we have some blue-type individuals. So, we define the state of the system.

by a single variable and that variable let's call it i and i represents the number of blue type individuals in the population. We will also assume that since this is a chemostat and it always operates at a steady state, the total population size in this chemostat always remains constant at n so what this what these two immediately automatically imply that at this instant in time number of b type individuals which is blue is i and number of a type individuals which is black is simply n minus i this is blue This is the state of the system. And of course, if i is equal to zero, then we are talking of a state where there are only black type individuals in the population.

If i is equal to one, then we are talking of this state where only one blue individual has come in the population and so on and so forth. I equal to N means that it is only blue type individuals which are there in the population. So, if you think about the transition probability, the probability that we are interested in finding out, we are interested in finding out that what is the probability that the system transitions from i equal to 1 to i equal to n. Because this is a state where there are n blue type individuals. This is a state where there is only one blue type individuals, which is the start of the journey for the blue individuals.

And we are interested in the system transitioning from i equal to 1 to i equal to n. That is our goal. So in a Moran process, we will assume not 1 or n. We will assume that the system is at state i. i could be anything. Different values of i will represent different stages of the process. We will also assume that the fitness of the two genotypes is rB and rA. These could be equal in the case when there are both equally fit or these could be unequal if B, if the blue individual is fitter than the black individual, then rB would be greater than rA.

Of course, we are not going to discuss the case where rB is less than rA because, as I said, selection will just get rid of the blue individuals that have come up in the population. Okay. So, in a Moran process, we are going to assume that since this is a chemostat, the total population size always remains n. This condition is never violated. So, if at any instant of time I were to take a snapshot of this chemostat and count the number of individuals present in it, the number would always be equal to n.

It never deviates. However, as you can see, this is a chemostat, so there are bacteria being continuously washed away from it, and there are new birth events continuously taking place in it. So, the total population size is always equal to n, but this is in the face of bacteria being washed away and bacteria being born. So, what this means is that death—which is bacteria being washed away—is equal to birth—which is divisions taking place. In the chemostat.

These two have to have to exactly balance each other. So, in a Moran process, we are going to we are interested in that.

Narrow sliver of time where one birth event happens. And which is exactly matched by. One death event. This is necessary so that the population size can remain at constant.

If only birth took place, then the population size would go to n plus 1, in which case this would be violated. Or if only death took place, then the population size would go to n minus 1. Again, that would be violated. So, the assumption here is that one birth... a new individual is born and one individual is washed away from the system at the same instant.

So the population size always remains constant. So if this is the case, one birth and one death is taking place simultaneously, we have four options. We could have a case where an A individual is born and B individual dies or washed away. We have another case where B individual is born and A individual dies.

We have two other cases: an A individual is born and an A individual dies. The last case is that a B individual is born and a B individual dies. These are the only four options for one birth and one death event in the Moran process when there are only two genotypes in the population. And according to our definition, these are the only two genotypes. So let us just note these here.

B represents the blue-type individuals whose fitness is rB and the state is I. So B is blue individuals. The state of the system is i. That means there are i number of B-type individuals and their fitness is rB. The corresponding quantities for the black individuals are: A is black individuals. Their number is n minus i and their fitness is A. So, what we are going to do is compute the probability associated with each of these four processes. Let's start with the first one. We are interested in the probability that a B individual is going to die—that the individual washed away from the reactor is going to be a B-type individual. If there are i individuals of type B out of a total of N, the chance that a B-type individual is going to get washed away is simply equal to i upon N. Because in a chemostat—let me draw the chemostat here again—

$$P(A\ birth, B\ death) = \frac{i}{n}\frac{(n-i)rA}{(n-i)rA + i\ rB}$$

$$P(A\ birth, A\ death) = \frac{n-i}{n}\frac{(n-i)rA}{(n-i)rA + i\ rB}$$

$$P(B\ birth, A\ death) = \frac{n-i}{n}\frac{(i)rB}{(n-i)rA + i\ rB}$$

$$P(B\ birth, B\ death) = \frac{i}{n}\frac{(i)rB}{(n-i)rA + i\ rB}$$

So, from the context of this death event, I have these individuals, which are N minus i, and I have the blue individuals, which are i. So, the probability of B dying is simply equal to i upon N. Because the larger i is—meaning there are more B-type individuals in the chemostat—the greater the chance that one of those i individuals happens to be here. If black is very large in number—meaning N minus i is very high and consequently i is very small—then the chance that a B-type individual gets washed away from the chemostat is going to be very small. Basically, what we are saying is that washing away is a completely random process. It could be any cell that happens to be close to the exit of the chemostat and is then washed away from the system.

Hence, the probability of death of a B type individual is simply proportional to its numbers and equal to I by N. So that gives me the probability of death. This has to be multiplied with the probability that an A type individual is born. This is computed, this is estimated with the following mathematical relationship. n minus I, I will write this first and then explain. N minus I divided by RA divided by N minus I times RA plus I times RB.

So, this is the probability that an A individual is born. In the numerator, In the numerator we have total fitness of A individuals, which is fitness of each A individual multiplied with number of A individuals that exist in the population. In the denominator we have total fitness of the entire population. the entire population so what this means is that if the if so this is this ratio here is the fractional fitness of the entire population which lies with the a type individuals if this number is very high

then there is a great chance that it's going to be an A type individual that is going to divide next. And if this number is low, then the chance that A type individual is going to divide next is going to be very small. So, this is again the chance that an A individual divides next is simply equal to total fitness of A individuals divided by total fitness of the entire population. This is just number of number of A type individuals times fitness of A individuals and in the denominator is number of A type individuals times the fitness of A individuals. plus number of B individuals times the fitness of B individuals.

So, these two multiplied with each other ensure that they give me an expression for the probability that in the next instant in the reactor, a B type individual is going to die and an A type individual is going to be born. This is the probability that a B individual dies. And this is the probability that an A individual is born. And I hope that makes sense to everybody. This is something that we are going to use quite often here on onwards.

One more thing before we move forward with this is that if this happens, then what happens to the state of the system? Remember the system was at state i, that means number of B-type individuals was i. If this happens, if a B-type individual dies and an A-type individual is born, that means the number of B-type individuals reduces by one. Because a B type individual died and number of A type individual increases by 1 because an A type individual divided hence their number went up by 1. So, the system transitioned from state i to i minus 1 where there were i number of B type individuals in the population prior to these two happening now there are only I minus 1 number of individuals left in the population which are of B type. So we're going to call this probability of transition from state i to state i plus 1.

I'm sorry, i minus 1. Because we started when there were I number of B type individuals. And after these two happened, I am left with I minus 1 B type individuals in the population. So I hope that makes sense to everybody. If this is clear, then the next three are going to take care of themselves and follow the same logic.

So let's do this. Let's do the next one. Get rid of this. So the next is the opposite of this: in this one, a B individual is born and an A individual dies, right? What that means is if these two things happen, the number of B individuals increases by one because it's a B-type individual that was born.

And the number of A-type individuals decreases by one because that's the individual that dies. Hence, after these two are done, the process. What we are computing is the probability that the system transitions from state i to i plus 1. Because after these two happen, the number of B-type individuals would have increased by 1. And that's equal to the probability that a B individual is born times the probability that an A individual dies.

We are going to use the same logic that we used in the above expression and compute the probability that an A individual is going to die, and that is simply going to be proportional to how many A individuals there are and how many B individuals there are. So, the probability of A dying is simply proportional to how many A individuals are there in the population, which at this stage is N minus i divided by N. And you should see this is the exact formula we wrote here because this was the probability of B dying, which was proportional to the number of B-type individuals. And now we are writing the probability that an A individual dies, which is proportional to the number of A-type individuals at this point in time. On the other hand, the probability that a B-type individual is going to be born is simply going to be equal to, using the same logic that we used in the previous expression.

The probability that a B-type individual is going to be born is simply the total fitness of B-type individuals in the numerator divided by the total fitness of the entire population in the denominator. So, the total fitness of B-type individuals is the number of B-type individuals, which is i, times the fitness of each B-type individual, which is RB, divided by the total fitness of the entire population. This is the total fitness of B-type individuals, i times rB, plus the total fitness of A-type individuals, which is n minus i times rA. What you should note here is that the denominator in these two is exactly the same. The denominator in the death terms is exactly the same.

Only the numerator changes depending on what probability we are calculating—which individual is going to die and which individual is going to survive. And this gives me the transition probability of the system going from state i to state i plus 1. So, that is 2 out of the 4 calculated, and we will continue these calculations in the next video and develop this theme a little further. Thank you.