Evolutionary Dynamics

Supreet Saini

Chemical Engineering

Indian Institute of Technology Bombay

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Lecture 37

I welcome everyone to this next video of the course. So, last time we set up these equations which were very similar to the ones we had done previously and our idea is to get the fixation probability of an adaptive mutation that confers a benefit to this new mutant that has come up in the population and in this video we will develop those equations while we did not actually solve the equations for the neutral case Because it is a relatively simple case mathematically, so those of you who are familiar with linear system or linear algebra should be able to solve those equations given all of that we discussed. But we also found two useful ways to think about this where we could solve the system without actually working through the equations. Those were by thinking of marbles in a jar game or just using intuitive logic to arrive at the solution that when a neutral mutant arises, its fixation probability is simply 1 upon n where n is the size of the population.

For an adaptive case, That intuition is very difficult to develop because this is a somewhat involved mathematical derivation. So, it is going to need some work. There is no intuitive analog that is present that we can use to simply arrive at the solution. And secondly, these adaptive mutations are the ones which are going to drive evolution of a population forward, which are going to lead to increase in fitness of these populations.

So we want to understand much more formally how these adaptive mutations fare in the population. So that's why the next task before us is actually working through those equations and solving them to arrive at the fixation probability, which is the critical variable that we are interested in in our representation is X1. The math is not complex. It is just very cumbersome because of the representations that we use. It's mostly just bookkeeping.

But if you go through with me through this bookkeeping, you will find that if we understand the representations and what each variable means, the math is just not very hard. So we can arrive at the answer fairly easily despite the cumbersome bookkeeping. That is the task before us that we will do in this and perhaps also the next video. So. To sort of define our goal, we have, as always, a situation where we have A type individuals.

Their number is, let's say, N minus 1. Their fitness is RA. And in this population arises a beneficial mutation whose number is 1 and fitness is RB. And this is in a chemostat. And we are interested in understanding what is the probability that the system transitions from this state to this particular state.

It is only B type individuals in the population that are left. All the A types have been eliminated given the fact that RB is bigger than RA. RB is greater than RA. What is the probability of this transition? And this is critical because this is how evolutionary change is going to be driven and fitness of population increase over time.

This process obviously doesn't stop here. Eventually, at some point when it's only blue-type individuals, the process will keep repeating itself. When another mutation, which is, let's say, one individual of fitness RC comes up, and this RC is bigger than RB. And now the red individual will outcompete the blue individual. So, these adaptive mutations are what drive the process of evolution forward, and hence we are most interested in them.

So, the same analysis is going to carry forward every single time, and hence we are going to pay particular attention to what is happening here. This is the probability that is of interest to us. In another context, we define this in the form of a number line where we have 0, n, 1, and we ask that if my system is present here, the number of this represents the number of p individuals in the system, and once the mutation happens, I am only going to have one of them. What is the probability that the system reaches state n or the B mutation reaches fixation.

So, we will keep this in mind that these are the bigger goals we are working towards while we do this somewhat bookie, patient, cumbersome math in this video. So, the equation that we have from the context of the number line is simply i i plus 1 i minus 1 beta i alpha i 1 minus alpha i minus beta i. And although we are saying that we started off from this point, we are going to discuss a general point I and not one, because as this mutant of our interest moves towards fixation, starting from point I equal to one, it's going to have to pass through every I, and only then it can reach N. So discussing it for a general I is very useful in that context. And we developed this equation last time: xi is simply going to be equal to beta i xi minus 1 plus 1 minus alpha i minus beta i into xi plus alpha i into xi plus 1.

We also know that x0 is equal to 0 and xn is equal to 1. That is our starting point. So this can also be written as a system of linear equations where we have these vector of unknowns, which is X naught, X1, X2 going all the way to Xn. This is equal to this matrix. times the vector x0, x1, x2, xn.

and this can be written as 1 0 0 all 0s, x 1 can be simply written as beta 1 times x naught and the term here will be 1 minus alpha 1 minus beta 1 and the term here will be alpha 1 and the rest will be 0s. and so on and so forth. And for the i-th term, this is going to be i minus 1 zeros and then beta i. Then the next term is going to be 1 minus alpha i minus beta i and then followed by alpha i and then again zeros. So, those of you who are familiar with linear algebra should be able to see that these n minus 1 equations are These two equations, collectively this is n plus 1 equations, can be simply written in this matrix form.

And this is akin to solving this problem where this is simply 0 and this is simply 1. So we can solve this system of linear equations whichever way. So now our task is to solve this system of linear equations and particularly find the value of x1. That is what we are going to be working towards. And what we will do towards that is we will define a variable.

So what I want is the variable x1. And it turns out that getting to x1 is not that easy, and hence I will define a variable called yi, which is just equal to xi minus xi minus 1. While it does have some sort of physical significance, we are going to think of yi as just some mathematical jugglery, some mathematical definitions that we did in order to arrive at our answer that we seek in as easy a system as possible. So what we are going to do is: If this is the definition of yi, then what is y1?

y1 is simply x1 minus x0. I know x0 happens to be 0. We know that from the way we have defined x's. What is y2? It is simply equal to x2 minus x1.

Both of which I do not know. y3 is simply x3 minus x2, and I can keep writing this till I get to yn, which is simply equal to xn minus xn minus 1. Let us write one more: yn minus 1 is going to be equal to xn minus 1 minus xn minus 2. So, we have So by defining a new variable yi like this, I can write the different values of y every time it takes a new value of i. So there are these n such values of y possible. And what I'm going to do is add all of them up.

So, on the left-hand side, I get y1 plus y2 plus y3 going all the way up to yn, which I can simply write as sigma i going from 1 to n yi. On the right-hand side, what you should notice is that when I begin to add these terms, The x1 here gets canceled with x1 because I am going to add. So, this is a negative x1, and this is positive x1. So, these two cancel.

Similarly, x2s cancel because in y2, x2 is positive, and in y3, x2 is negative. Similarly, x3 will cancel because in y3, x3 is positive, but in y4, x3 is going to be negative. So, everything cancels out. Nothing remains here except for the first term of the last yn and the second term of the first y1. So, this comes out to be xn minus x0.

But it turns out that I know both of them. x0 is simply 0, and xn is simply 1. So, sigma yi is equal to 1. That is something that is going to prove useful in a little bit. So, I have sigma yi, i going from 1 to n, yi is equal to 1.

$$y_i = x_i - x_{i-1}$$

$$\sum y_i = 1$$

And I will define here that this definition of yi is primarily for mathematical ease. It does not have any great physical significance in the process that we are studying. So, sigma yi is equal to 1. Now, let us go back to the recursive equation that we wrote. That equation said xi is equal to beta i x i minus 1 plus 1 minus alpha i minus beta i

times xi plus alpha i times xi plus 1. What I will do is I will open this bracket up, which will give me three terms. So, I get xi equals beta i xi minus 1 plus xi minus alpha i xi minus beta i xi plus alpha i xi plus 1. This is just simple math with what we already had so far.

So, these two cancel. There is xi on both sides. So, I can just take one over to the other side, and that cancels. And what I will do is I will take the beta terms on the other side. So, I will take this term and this term on the other side.

And then the resulting equation becomes beta i xi because this term becomes positive minus xi minus 1 because this term when it goes there becomes negative this term becomes positive and hence I get this and what is left on the right hand side is alpha i xi plus this term is positive so xi plus 1 minus xi. What this gives me is x i plus 1 minus x i is equal to beta i upon alpha i times x i minus x i minus 1. So this is what simply rewriting the equation that we have always had. We have had this equation for a few videos now and simply rewriting it we get this particular form.

But what we should realize here is in the previous slide when we defined this variable yi that was just xi minus xi minus 1. So the definition of yi was just xi minus xi minus 1. This was how I defined yi. Definition of yi from the previous slide which means I can write this as yi plus 1 is equal to beta i upon alpha i times yi and now I get this

So, let me call this ratio of beta i upon alpha i as gamma i. So the two consecutive y i's are connected to each other by this ratio gamma i. And that is where that is what I meant when I said that there is a lot of bookkeeping involved in this. But the math is not difficult math. It's just a lot of terms that are introduced. So there is this gamma i which connects y i to y i plus one.

But I also know that sigma y_i, if I add all the y_i's, I get one. I got that from the previous slide. So, we'll start from the first definition of Ys. Y1. So, if we start defining the Ys, we get

$$\gamma_i = \frac{y_{i+1}}{y_i}$$

So, first of all, we have an equation that y_i+1 is equal to gamma_i times y_i. We know that y1 is simply equal to x1 minus x0. That comes from the definition of y_i, but x0 is simply 0. That we have known for some time. So, the first result I get is y1 is equal to x1.

Secondly, I get, now I will apply this where I get y2 is simply equal to gamma_1 times y1, which means, but y1 is just x1. So, I get gamma_1 x1. So, this means y2 is just equal to gamma_1 times x1. If I apply this for i equal to 3, I get y3 is equal to gamma_2 y2. But I know y2 is simply gamma_1 x1.

So, I get y3 as gamma 2 gamma 1 x1. So, I get y3 as gamma 2 gamma 1 x1. And so on and so forth, I can keep on going. Maybe let us do one more. We get y4 as equal to gamma 3 y3, but y3 is just equal to gamma 2 gamma 1 x1.

So, I get y4 is equal to gamma 3 gamma 2 gamma 1 x1. And hence, y4 is simply equal to gamma 3 gamma 2 gamma 1 x1, and so on and so forth. I will keep on doing this till I get to yn, which will simply be equal to gamma n minus 1 gamma n minus 2, going all the way down to gamma 1 times x1, and that I am going to write here and simply get yn is equal to gamma n minus 1 gamma n minus 2. Going all the way down to gamma 1 times x1. So that is the expression that I have for the different y's.

But now I am going to combine this. With the fact that I know from the previous slide that sigma yi, if I were to add all the yi's from 1 to n, sigma yi is simply equal to 1. We can go back and check this, and that was when we added all these together, we got all these terms canceled out, and sigma yi was simply equal to xn minus x0, which was equal to 1. So when I look at this analysis here, I want to add these two—I want to add not these two, but all these n expressions. And on the left-hand side is a fairly straightforward thing, which is sigma yi.

This I don't have to do anything because I know when yi is summed from 1 to n, this is just equal to 1. The right hand side, I can get X1 out and I see that I have eliminated every other variable from this analysis and I'm only left with X1. And that was the entire goal of doing this exercise that I want to get an expression for the variable X1. So I get x1 if I take x1 common in the first term there is 1 in the second one there is gamma 1 then gamma 2 gamma 1 then all the way to gamma n minus 1 all the way to gamma 1 and that is x1. This is the equation that I have been seeking all along that now I have this equation which relates x1 and gives it to all other variables x1, x2 everything has been removed and I get an expression for x1.

So, let me rewrite this again that 1 is equal to the expression that I get is 1 is equal to x1 times 1 plus gamma 1 plus gamma 2 gamma 1 plus gamma 3 gamma 2 gamma 1 all the way going up to gamma n minus 1, gamma n minus 2 times gamma 1. This tells me that the value of x1 is simply equal to 1 upon this entire expression which is 1 plus gamma 1 plus let me rewrite it in this form gamma 1 gamma 2 gamma 3 going all the way to gamma 1 gamma 2 gamma n minus 1 and that is the expression for x1. And remember X1 is the fixation probability that the system is at state 1 and X1 is the probability that the system reaches this particular state.

When there is starting from a place where there is only one type of the B kind of individual present in the population. So we are getting somewhere here. But now the problem is that this is still difficult to use because I don't have these expressions for these various gammas. Remember that we defined gamma a few slides ago. when we were trying to relate yi plus 1 with yi, gamma i was defined as beta i divided by alpha i. So,

Gamma I is defined as beta I divided by alpha I, where alpha and beta were these probabilities that when I am at a state I, when I am at state I in the system, I have three fates. Their relative probabilities are beta I is the probability that I will go back from I to I minus 1. Alpha i is the probability that I will go forward from i to i plus 1 and 1 minus beta i minus alpha i is the probability that I will remain at i. So, it is this ratio of these two probabilities of beta i is the probability of moving back and alpha i is the probability of moving forward. That ratio is gamma i and it turns out that now I can derive a value of x1

But only if I know the value of gamma for all possible values of I. I need to know gamma 1, I need to know gamma 1, gamma 2, 3 and all the way up to gamma n minus 1. So, that is the next task before us that we have to find out all these values in order to proceed further and get the fixation probability associated with a beneficial mutation that has arisen. And that is what we will continue with in the next video. Thank you.