Spatial Statistics and Spatial Econometrics
Prof. Gaurav Arora
Department of Social Sciences and Humanities
Indraprastha Institute of Information Technology, Delhi

Lecture - 12B
Variogram and Semivariogram

All right. So, welcome back to the second part of lecture 12, now we are going to sort of you know, formally define a model and provide a model for the Variogram, how do we sort of conceptualize it mathematically when we have data?

So, on the right top corner of your screen, you see an origin 0 comma 0 and two locations given by vector s and vector s plus h ok, h is; obviously, a lag vector right? And when I talk about like you know location s it has two coordinates x and y, right? it has coordinates s_x comma s_y, sorry about that ok, we will correct it in a minute ok, It is s_x comma s_y right, and s plus h is again you can think of these as coordinates s tilde x and s tilde y alright?

Now, when we say, you know, s_x comma s_y or x comma y we are simply joining a vector from the origin which goes s_x units horizontally and s_y units vertically right? And we are trying to understand spatial dependence or spatial contiguity measures between points s and s plus h right? s plus h could also be defined as S_2 right? So, s could be S_1 the point you know in the South is S_1, and in the North is S_2 as marked earlier h is accounting for both the distance between points S_1 and S_2 in space and also the direction. The direction is that it is closer to a North-West direction moving from S_1 to S_2 ok.

So, having known that what we have is the definition of variogram that is we have the following that variance of Z of S_1, S_1 is a vector minus Z of S_2 is given as 2 gamma as a function of the distance between vectors S_1 and S_2 given as the L_2 norm ok. This is true for all S_1 and S_2 that belong to the domain D and domain D is a stationary domain you know without a doubt ok.

Now, the quantity 2 gamma S_1 minus S_2 is called a variogram no surprise is there, right? it is a function of S_1 minus S_2 which is the definition of this vector h which is the lag vector; gamma is by itself called a semivariogram right by convention. And it is also known as a "structural function" in meteorology, fluid mechanics, and so on right.

So, if they are students of meteorology taking this course or fluid mechanics taking this course they may have come across something called a structural function. Well, it is nothing, but the semivariogram that we are studying here right? It is known as "mean squared difference" right? It is this is quite a literal translation and if you have heard of this you are most likely coming from, you know, a background where you have used time series quite a bit right?

So, if you are or have been a time series analyst you must have come across this mean squared distance right and this mean square distance is the analog of the semivariogram in space right.

Of course, we know that there is a difference between time series data and spatial data time series data is unidirectional it does not go in all directions possible right? So, it is not a multi-directional process, it is a unidirectional process, also it is you know, you have discrete time points in space, right?

So, there is between t_1 and t_2, t_1 and a half does not really become a unit of analysis. Whereas in the case of spatial data between t_1 and t_2, t_1 and a half t_1 third, you know, t_1 plus t_1 third or t_1 plus t_2 by 2 these are all you know legitimate you know spatial entities ok. Now what we are going to do going forward is we are going to study the properties of a variogram.

So, the first property is that notice that gamma of h is going to be equal to gamma of minus h ok. What does that mean? Gamma of h being is equal to gamma of minus h which means that the measure of spatial contiguity that is derived from this device called semi-variogram is going to be the same if you were to start from S_1 and go to S_2 or start from S_2 and come to S_1, it will not matter to this measure of spatial dependence right? and it should not right.

It is spatial dependence between S_1 and S_2, it does not matter if I am going from S_1 to S_2 or S coming from S_2 to S_1 right? Where h is nothing but S_1 minus S_2 and minus h would be S_2 minus S_1. So, the L_2 norm of S_1 minus S_2 is mathematically the same entity as the L_2 norm of S_2 minus S_1 ok.

Second, we have gamma of 0, gamma of 0 is equal to 0. So, if there is no distance right? So, we have looked at the same entity so the variance of ZS_1 minus itself will be exactly 0 right?, and very very critically and as h approaches 0, you know, we may have gamma h equals C_0

which is a non-zero entity that is non-zero. This C_0 is called the nugget effect it is known as the Nugget effect.

And it captures what is called the micro-scale variation in spatial data in spatial data what this means is that if S_2 were very very close if S_2 were exactly S_1 gamma is 0. If S_2 is very very close and it is approaching S_1, in the sense that h is approaching 0 right? if h is approaching 0 then you know, if we have some kind of a micro-scale variation it is called a nugget effect.

The term nugget comes from the gold mine. So, when you know gold sort of exists as these nuggets right, these are the smallest possible entities that exist in space which would mean that a nugget and its core or nugget is large enough that you would have to move slightly farther away to be able to sort of get the smallest possible entity from the core right.

It is a nugget you can look it up right and this nugget has characteristically different variation properties than the coal that exists around itself right? So, these nuggets are distinctly different entities that from which the coal you know in which gold occurs beneath the surface right? So, the term nugget comes from the gold mining process and of course, spatial statistics; that means, it is genesis sort of happens from you know has come from there right?

These micro-scale variations are kind of distinct enough that we have coined a separate term for them right? So, let us sort of move forward with the properties, and while we do that we will be able to sort of understand you know, a few more properties of the nugget effect. So, I am going to write this down again on the next page, so that we can have continuity in the discussion.

So, we said that property 2 was that gamma 0 is 0 and as h approaches 0 we have that gamma of h approaches C_0 and this C_0 is called the nugget effect. Now note that C_0 could be a measurement error right? So, it may not be in fact, a micro-scale variation it may be a measurement error when in fact, I mean after all we are closing down to a location in space beneath the surface and trying and measure and define micro-scale variations right.

So, they may not be a real thing, in practice, we cannot measure them, right? In practice what happens or practically what happens is we have data which is let us say ZS_i such that i is location 1, location 2, location 3, all the way till location n right let us say these are our data and we usually sort of you know and usually we do not have the data availability.

The available data do not provide us an opportunity to comment upon you know the situation where h is indeed very very small. You have to understand that to be able to measure micro-scale variation we need to have data sets where samples over space are you know collected close to each other. We have seen many examples till now and what we know is that it probably never really happens.

So, we never have data with a lag approaching 0. So, if we have no data there where lag is not approaching 0 we cannot practically measure micro-scale variation; that means, that means usually right. I am not going to say this is an exhaustive case, but usually, we cannot evaluate or estimate C_0 using real-world data sets. And now so, of course, now we know that you know C_0 which is a nugget effect is a physical real-world phenomenon.

It is a phenomenon we may not be able to do practically we may not have the device to do that, but it is indeed a phenomenon, right? So, what do we do right how do we go around evaluating or estimating C_0? So, to that effect, a workaround was provided or offered by Matheron in 1962, what is the workaround? So, Matheron suggested that we specify C_0 as C_{MS} plus C_{ME}. Now, so basically he suggested that we decompose the micro-scale variation in two parameters; C_{MS} which is a constant, and also C_{ME}.

Now, C_{ME} without a surprise is you know due to measurement error right; that means, this variable will be completely random, this is out of our control, and it is happening due to exogenous reasons that we as analysts cannot control at all right, anything about it nothing about these is systemic they are completely random processes. So, I am going to say fully random.

C_{MS} is the negative effect or micro-scale variation right? So, I am going to say micro-scale variation that is modeled as a white noise process. What does it mean when I say that C_{MS} is going to be modeled as a white noise process, all I mean is that mean that is C_{MS} is distributed normally with the lower sort of bound greater than 0.

So, you know, it is always greater than 0 and it could be as large as infinity. So, it is a positive number, of course, we are looking at a variation measure, we do not have a negative variation measure. So, we have positive variation measures that variance is a second-moment property. So, C_0 is a second-moment property, right? So, it cannot be a negative number. So, it is truncated it is not fully random it cannot go from minus infinity to plus infinity, but it is quite random, right?

And it is a truncated normal distribution with mean mu MS and variance sigma squared MS right? So, I can visualize that C_{MS} will be a positive number and it will sort of follow a distribution that looks like the normal half-bell curve on the right-hand side with the 0 not included. So, this is normally truncated normal, sorry about that. Truncated normal with mean mu MS and variance sigma squared MS.

Now, in most practical applications C_0 is predicted from given data, and C_{ME} is assumed to be 0 right. So, I am just going to leave a note here that in most applications in most applications, C_0 is "predicted". So, we are going to conduct prediction on unknown values right we talked about prediction meaning we are going to exploit the dependent spatial contiguity structure and try and tell what is happening in the micro-scale neighborhood.

So, we are going to conduct prediction and not estimation by observations right? So, we are going to conduct prediction for C_0 from the given data, and in most cases, we are going to assume measurement error C_{ME} to be just 0 ok. Unless we have some information about where the measurement error could arise from right?

Some of my ongoing research suggests that for example, in groundwater monitoring the measurement error indeed arises due to things like human reporting. So, there is a monitoring well, and human beings are going on to the well and manually note down the depth of the groundwater level. There could be human error, right? there could be measurement errors because the well got choked right? There might be silting and you know, the water cannot flow through.

So, whatever you are seeing is not a measure of reality because the water either could not flow out or water could not flow in to reflect the levels around this well right? The wells could even sort of you know become closed because for certain reasons right?

So, the measurement errors can be systematically directional you know something that we can understand. But usually, you know they are assumed to be 0. So, unless we have information about where measurement error may arise from we will assume it to be 0. So, moving forward now we are looking at properties. So, we will look at the third property after this.

So, the third property is that, if the variogram 2 gamma is continuous it is a continuous function at the origin. If it is continuous at the origin then we say that Z which is our spatial random process is L_2 continuous, what does L2 continuous mean? So, I am just going to say meaning that the expectation of ZS_1 minus ZS_2 squared. So, it is the expectation of the square of the first difference, right? So, the mean value of the first difference, the first squared difference mean value of squared difference right?

ZS_1 minus ZS_2 will approach 0 at the origin that is h when h goes to 0, when h goes to 0 you have a situation where the variogram value will start to diminish that is the expectation of the difference squared value between observations at any two locations will approach to 0. So, then gamma is a continuous function, right? it is a continuous function at the origin remember we are talking about the origin right?

We can have the converse being true that is if 2 gamma does not approach 0 at the origin; that means, it is test continuous at the origin. If it is discontinuous if 2 gamma is a variogram is discontinuous at the origin then we say that Z is not L_2 continuous right; that means, that this condition that expectation ZS_1 minus ZS_2 squared will not approach 0 as h approaches 0 right?

Such a process is considered to be a spatially irregular process so or irregular spatial process ok. Now, this discontinuity at the origin is called or known as you would have guessed by now the nugget effect ok. I mean; obviously, have already known we already know the notation that it is C_0 ok.

The next property is that if 2 gamma h is a positive constant ok. If 2 gamma h is a positive constant of course, we are going to talk about a positive constant except at the origin, at the origin, you know we expect it to be either 0 or a micro-scale variation measured by the negative effect that we have talked about earlier. If 2 gamma h is a positive constant that is 2 gamma h is some K constant, and K is not a function of h.

So, now gamma h does not vary by h it does not matter what is the distance between two locations it is just a constant. Now, what do we expect then you know the variogram representing, then you know Z of S_1 and Z of S_2 that is the values at two different locations are spatially uncorrelated. I mean for all S_1 and not equal to S_2 regardless of how close these two points are in space.

These two points or locations right so, when I say these two locations I mean S_1 and S_2 are in space right? If the variogram value does not vary by h then what you are looking at is a spatially random process right? you are going to start to look at the left-hand side panels that we looked at you know as part of lecture 12 in the previous Part-A of it right?

So, that is why the variogram provides us with a measure of spatial dependence or spatial contiguity over space right? it is a legitimate measure in that spirit. So, let us move forward we have to look at one more property of the variogram you know before we sort of move on to the covariogram and the correlogram ok.

So, if you say 2 gamma S_1 minus S_2, I am going to say the L_2 norm which is nothing but you know this is equivalent to 2 gamma h right? This is equal to 2 gamma naught S_1 minus S_2 that is now we have the distance as you know multiplier to a constant 2 gamma naught right?

So, this gamma naught is a constant, this is like saying you know I could say this is just 2 k h ok where k is a constant right? Then what are we looking at we say that 2 gamma which is the radiogram that we have estimated or figured evaluated from the data is called an isotropic variogram.

Such a variogram sort of you know represents what we call an isotropic spatial process. Now, the antonym of an isotropic spatial process is an anisotropic spatial process, if you come from physical natural sciences earth sciences, geology etcetera you would understand what isotropic and anisotropic processes are perhaps from natural sciences, and physical sciences you will understand it better. For those who have not heard of you know isotropic versus anisotropic situations.

Let us look at a figure and see what an anisotropic process is. So, in this figure on the left what you see is a schematic of a factory over space which has a smoke plume, and of course, you know because the factory sort of goes on to sort of produce whatever it produces it also produces smoke as a byproduct which is a social bad right.

Pollution causes health problems you know visibility problems, and it is considered a social bad you know typically. Now, the distribution of pollution over space is something that can be modeled using a radiogram or different spatial dependence devices, everything that we have studied about groundwater is also true for pollution or pollutants above the ground.

The only difference is we do not see them either right just like we do not see groundwater beneath the surface, we cannot see pollutants typically we cannot see of course, in Delhi if you are in the winter season and you have smog of course, you can see pollution with your eyes, but you cannot tell apart where that the smog density is you know changing is the gradient of course if it is very high or it is too small right?

So, either visibility is clear or it is unclear or it may be really bad. So, we have this discrete understanding of data over space you know using our visual interpretation. But to measure pollution you will need devices you will have to install air pollution monitoring devices and depending on where you install these devices you are going to sort of able to get a sample of different levels of air pollution right?

In this realistic real or real-world situation what happens is that the way pollution disperses over space depends on the direction of the wind right. If the direction of the wind is from east to west you will have higher dispersion a carryover effect of the wind in the east-to-west direction. If the wind is very sort of strong then you will have a quicker dispersion of pollution, if the wind is not so strong or wind systems are weak that day you are going to have less dispersion and a higher propensity of observing a smog situation right?

But what this wind is doing, is giving a structure to the way spatial data are going to be you know distributed in space right? So, if you look at the figure that is down below, what you see here is the center of a point where pollution was being emitted. So, this is the plume right and from the plume and in the direction that is opposite to the direction of the wind, the dispersion is quite small or it is happening in over a small area land area or surface area than in the direction of the wind right?

So, of course, the wind is going in the East direction. So, you have a higher sort of dispersion to a higher stronger degree of dispersion in the Eastward direction than you see in the West direction in fact, also in the North and the South directions. Although in North and South directions you cannot differentiate between them right?

So, this process is isotropic in the North-South direction you can sort to understand the distinction between North and South and if we had a similar thing going on in the East-West direction as well we would not be looking at an ellipse rather we will be looking at circles right? we will be looking at concentric circles. So, if dispersion happens in concentric circles that are no direction and is favored against the other those processes are called isotropic processes.

And the process that you see on your screen where a certain direction any one direction in the two-dimensional space has a different process of dispersion or different process of evolution of spatial data than the other, then you have what is called an anisotropic process. You can imagine a similar process going on with you know groundwater recharge beneath the ground. Well, when ground water sort of you know when precipitation happens and water enters the ground what is happening is gravity is pulling the water down.

And there is going to be anisotropy because of the gravitational process of gravitational pull you know towards the ground. So, if you are measuring if your interest is in spatial statistics that are lateral in nature well and good, but if you are studying or measuring spatial processes that are vertical in nature beneath the ground then perhaps one should be worried about anisotropy.

And in those situations, we have to separately estimate or evaluate or study a variogram in you know the East-West direction and then separately in the North-South direction in the direction of which is the origin of anisotropy ok. So, anisotropy is a physical process it complicates the situation for you know spatial statisticians and spatial econometricians so analysts like ourselves or data analysts.

However, for this course when we study variogram, the variogram estimation, evaluation, and calculation, we will focus solely on isotropic processes. So, I am not going to as part of this course co over those situations you know because the focus of this course is more on also econometrics which is to say that we are going to talk about social data and econometric data.

And you know with social data I am not sure, but you know I have not to sort of, I have not seen applications that deal with anisotropic data right? Although I am not ruling them out, the scope is small enough that in this course we are going to keep those out of our discussions.

So, we are going to sort of you know end this part here. And in the next part, we are going to spend some time perhaps it will be a short lecture on the covariogram and the correlogram ok. So, see you in the next part.

Thank you for your attention.