Spatial Statistics and Spatial Econometrics
Prof. Gaurav Arora
Department of Social Sciences and Humanities
Indraprastha Institute of Information Technology, Delhi

Lecture - 17B
Spatial Dependence in a Regression Model

Welcome back to the second part of lecture 17. In the first part, we established a spatial dependent specification in the R regression model. And then, we came to a point where we wanted to evaluate the impact of this spatial dependence, as specified through the variance-covariance structure of the error model on the least squares estimators, right?

$Next\ Step$: $Evaluate\ the\ implication\ of\ spatial\ dependence\ in\ model\ errors\ on$

$LS\ estimators$?

$$P\left(\vec{S}_n\right) = \beta_l + \beta_2 R\left(\vec{S}_n\right) \quad + \ \delta\left(\vec{S}_n\right)$$

$$Cov\left(\delta\left(\vec{u}\right),\ \delta\left(\vec{v}\right)\right) = \sigma^2 \rho^{\|\vec{u}-\vec{v}\|} \quad ;\ \vec{S}_n \epsilon \left\{\vec{S}_1, \vec{S}_2, \vec{S}_3 \vec{S}_4 \ldots, \vec{S}_N\right\} \equiv DcR^2$$

$$\hat{\beta}_{2,LS} = \frac{\sum_{n=1}^{N}\left(P\left(\vec{S}_n\right)-\bar{P}\right)\left(R\left(\vec{S}_n\right)-\bar{R}\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2} \quad ;\ V\left(\hat{\beta}_{2,LS}\right) = \frac{\sigma^2}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2}$$

$\rightarrow We\ learnt\ that\ when\ A1\ -\ A6\ hold\ \hat{\beta}_{2,LS}\ is\ B\ L\ U\ E.$

$But\ A3\ is\ violated\ under\ spatial\ dependence\ of\ model\ errors.$

$min\ variance$

$E(\hat{\beta}_{2,LS}|R) \neq \beta_2$

So, we figured that the least squares estimators look like what we see on your screen here. So, we were working on a model of housing prices and we are modeling those as a function of observed data on the spaciousness of a house at a given location Sn, which we index by the number of rooms, right? But we really want to understand how spacious this property about which we are trying to understand the pricing.

So, we figured that we will get an estimate of this coefficient beta 2 based on the least squares algorithm which is beta hat 2 least squares LS which is equal to the covariance between P and R divided by the variance of R. We also figured that this beta hat 2 o LS is also a random variable by itself. And so you know what we also have to report is its precision, its precision metric which is the variance of beta hat 2 LS, right?

What we also know from our previous lecture that is lecture 16 is that under the classical assumptions, a least squares estimator is blue. When I say it is blue, I want to say that it is the best linear unbiased estimator. Now, these are the properties of this beta hat 2 LS under the assumptions A 1 to A 6. But we know that A 3 is violated under spatial dependence of model errors, correct?

So, now how do we go about evaluating these properties? Well, beta 2 hat o LS or LS is going to be still an estimator. It will not be a non-estimator just because there is spatial dependence. It is still an estimator. It is also still a linear estimator. Remember, it is linear in P. So, if the coefficient parameter is linear in P which is linear in P; that means, that it is still a linear estimator, right?

The question now will arise, is this estimator unbiased and is it best? Is it the best estimator? So, we understand what unbiased means. So, basically what we are asking is how the expectation of beta 2 hat LS given the data on R compares with the true value beta 2. And second best means that this estimator is the one which will have the minimum variance upon all the estimators of beta 2, right?

Now, we have to evaluate whether or not both these properties hold when A 3 falls apart. So, let us go out and evaluate, whether or not the two properties best and unbiased you know still stay afloat when we introduce spatial dependence in the model errors that we relax assumption A 3. So, let us do that.

$Is\ \hat{\beta}_{2,LS}\ unbiased\ when\ \delta(.)\ ehhibit\ spatial\ dependence?$

$$E(\hat{\beta}_{2,LS}|R) = \beta_2$$

$$L.H.S:\quad E(\hat{\beta}_{2,LS}|R) = E_R\left(\frac{\sum_{n=1}^{N}\left(P\left(\vec{S}_n\right)-\bar{P}\right)\left(R\left(\vec{S}_n\right)-\bar{R}\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2}\right)$$

$$= E_R$$

So, what we are saying is we are asking is beta hat 2 LS is unbiased when delta S_n or let us say I do not want to complicate the notation when deltas that is the model errors exhibit spatial dependence. This is the question. So, what we have asked is that does the expectation of beta 2 hat o LS given the data R, on R equal to beta 2, where beta 2 is the true value which is the population model, right? It is the truth that we are after. We do not ever observe the truth.

But we are still able to sort of figure out whether or not, these things, you know what we have gotten from a data-driven estimate is close enough or not to the truth. So, I am going to write down work with the LHS. I have to work with the LHS. So, I have an expectation of beta hat 2 LS given R which is to say that I am working with an expectation of you know, So, just to make my notation easier, I am going to just say expectation conditional R is E sub-R, this means I am working with conditional expectations.

In conditional expectation R becomes the constant, right? So, we are going to treat R as a constant value when we apply this expectation operator to the formulation that we are going to just write, right now. n equals 1 to N, P S_n minus P bar times R minus R bar. Remember, everything that is R, is R S_n or R bar, they are both constants with respect to this expectation operator; R, R bar the whole square.

Now, we know that this P S_n through my regression model is beta 1 plus beta 2, R S_n plus delta Sn. So, the only thing that is random inside the expectation operator is going to be indeed the delta because R is constant. We are looking at conditional expectations. Beta 0 and beta 1 are both constants, they are just true values of these parameters. They are constant values, right?

So, the only thing to look out for is delta S_n. So, I am going to just say that you know you can write down, you can do the mathematical manipulation, and you can show that this expectation, the stuff inside the expectation operator will reduce to the following.

$Is\ \hat{\beta}_{2,LS}\ unbiased\ when\ \delta(.)\ ehhibit\ spatial\ dependence?$

$E(\hat{\beta}_{2,LS}|R) = \beta_2$

$$L.H.S: \quad E(\hat{\beta}_{2,LS}|R) = E_R\left(\frac{\sum_{n=1}^{N}\left(P\left(\vec{S}_n\right)-\bar{P}\right)\left(R\left(\vec{S}_n\right)-\bar{R}\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2}\right)$$

$$= \beta_2 + \frac{\sum_{n=1}^{N} E\left(R\left(\vec{S}_n\right).\ \delta\left(\vec{S}_n\right)\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2}$$

$$where,\ E\left(R\left(\vec{S}_n\right).\ \delta\left(\vec{S}_n\right)\right) = Cov(R|\vec{S}_n), \delta\left(\vec{S}_n\right) \equiv E(\delta(.)|R(.))$$

$$= 0\ by\ A2\ (refer\ to\ Lecture\ 16)$$

$$E(\hat{\beta}_{2,LS}|R) = \beta_2 \Rightarrow \hat{\beta}_{1,LS} = \overline{P} - \hat{\beta}_{2,LS}\overline{R}\ is\ also\ unbiased.$$

Result:

Least squares estimator is unbiased in presence of spatial dependence in model errors

(*refer only A*3)

i.e., a form of heteroscedastic errors.

So, we will have expectation beta 2 hat LS is equal to beta 2 plus summation n equals 1 to N expectation R S_n times delta Sn divided by n equals 1 to N, R S_n times minus R bar the whole squared. So, what is interesting now to us is this expectation of the product of my explanatory variable R and the error term delta, right? This term is nothing, but a representation of the covariance between R S_n and delta S_n.

And this covariance is nothing, but a representation of the expectation of delta given R, which is 0 by the second assumption, right? So, the second assumption suggested that the conditional expectation of errors is conditional on the x which is, right when I say conditional expectation it is equal to 0.

So, A 2 is not relaxed, and A 3 is relaxed, right? So, when I say A 2 I would always say refer to lecture 16. So, lecture 16 as you can see is very important, it is very critical, right? So, what will happen is that this second term will vanish, right? So, that means, this second term will vanish and I am left with beta 2. So, the expectation of beta hat 2 LS given the data on R is just the true value beta 2. That means, very important that the least squares estimator is unbiased in the presence of spatial dependence in model errors, right?

Specifically, I am only relaxing A 3. So, you know we will see that it can have more complicated implications. But for now, so far as spatial dependence is concerned we are only relaxing A 3.

So, even when A 3 is relaxed, beta hat 2 LS is unbiased. It is a good guess of you know the true beta 2 value. So, and by extension, this will mean sorry, I will use a different pen here. This will imply that beta 1 hat LS which is nothing, but the P bar minus beta 2 hat LS times R bar is also unbiased.

So, the regression estimators are unbiased in the presence of spatial dependence. This is a form of heteroscedasticity in data. So, we said that A 3 ensures homoscedastic errors, right? So, we are working with a form of heteroscedastic error. Again, you should go back and read Wooldridge's book if you have not heard of this term before.

Although, I believe that you know having a more general variance-covariance matrix, meaning that it is heteroscedastic and non-homoscedastic, non-spherical errors is a sufficient explanation. But for details please refer to Wooldridge's book, right?

So, we have this very important result that you know, even when we have the spatial dependence going on least squares which is seemingly simplistic, is still does a very good job. I mean I have a very good guess of the point estimate. My point estimate is a very good guess of what is happening in the reality. So, going back and seeing saying, you know I have figured that I am also going to have unbiasedness, so far as the presence of spatial dependence is concerned which relaxes assumption A 3.

$Is\ \hat{\beta}_{2,LS}\ the\ B\ E\ S\ T\ (minimum\ variance)\ estimator\ of\ \beta_2\ when\ Cov\left(\delta\left(\vec{u}\right),\ \delta\left(\vec{v}\right)\right)\neq 0\ for\ some\ \vec{u}\neq\vec{v}$

$$V(\hat{\beta}_{2,LS}|R) \equiv V_R\left(\hat{\beta}_{2,LS}\right) = V_R\left(\frac{\sum_{n=1}^{N}\left(P\left(\vec{S}_n\right)-\bar{P}\right)\left(R\left(\vec{S}_n\right)-\bar{R}\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2}\right)$$

$$= V_R\left(\frac{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right).\ P\left(\vec{S}_n\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2}\right) = \frac{1}{\left[\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2\right]^2} V_R\left(\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right).P\left(\vec{S}_n\right)\right)$$

$$P\left(\vec{S}_n\right) = \beta_1 + \beta_2(R\left(\vec{S}_n\right) + \delta\left(\vec{S}_n\right)$$

$= V_R$

So, as a next step we are going to now ask whether we are going to ask is beta hat 2 LS is the best. When I say best, I mean minimum variance, right? The best estimator of beta 2 is in the presence or let us just simplify this and say when covariance delta u delta V is nonzero for every or some not every; for some locations location pairs u not equal to V, right?

So, some pairs of locations that are separated by a distance or let us say a lag, a spatial lag, we have you know do we still have a best estimator? So, now, for that, we have to worry about beta hat 2 o LS. So, I am going to look at the variance of beta hat two least squares, given data on R which I am now going to define as V_R, just for concise notation beta hat LS which is equal to the variance of R; not variance of R variance conditional variance operator on to i equals 1 to N.

So, I am just going to write the formulation of beta hat 2. So, this is going to be P S_n minus P bar R S_n minus R bar divided by summation i equals 1 to N, R S_n minus R bar the whole square, right? This can be then shown to be equal to summation n equals 1 to N, R S_n minus R bar times P S_n divided by summation n equals 1 to N, R S_n minus R bar the squared.

Now, I have said that I am taking this variance conditional on R. So, that means, R is a constant. So, the denominator will directly come out, right? It is a constant. So, when I am with constant comes out of a variance operator it comes out as a whole square. So, it is going to come out as 1 over summation n equals 1 to N, R S_n minus R bar whole the thing squared, the variance of summation n equals 1 to capital N, R S_n minus R whole thing times P S_n. So, it is all in the denominator.

Now, we had said just like in the case of unbiasedness when you were evaluating it, we will substitute P S_n with beta 1 plus beta 2 R plus the error delta.

$Is\ \hat{\beta}_{2,LS}\ the\ B\ E\ S\ T\ (minimum\ variance) estimator\ of\ \beta_2\ when$

$Cov\left(\delta\left(\vec{u}\right),\ \delta\left(\vec{v}\right)\right) \neq 0\ for\ \vec{u} \neq \vec{v}$

$$V(\hat{\beta}_{2,LS}|R) \equiv V_R\left(\hat{\beta}_{2,LS}\right) = V_R\left(\frac{\sum_{n=1}^{N}\left(P\left(\vec{S}_n\right)-\bar{P}\right)\left(R\left(\vec{S}_n\right)-\bar{R}\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\bar{R}\right)^2}\right)$$

$$= V_R\left(\frac{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\overline{R}\right).\ P\left(\vec{S}_n\right)}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\overline{R}\right)^2}\right) = \frac{1}{\left[\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\overline{R}\right)^2\right]^2} V_R\left(\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\overline{R}\right).P\left(\vec{S}_n\right)\right)$$

$$P\left(\vec{S}_n\right) = \beta_1 + \beta_2(R\left(\vec{S}_n\right) + \delta\left(\vec{S}_n\right)$$

$$= \left(\frac{1}{\left[\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\overline{R}\right)^2\right]^2}\right) V_R\left[\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\overline{R}\right)\delta\left(\vec{S}_n\right)\right]$$

$$V_R\left[\sum_{n=1}^{N=2} A_n\delta_n\right] = A_1^2V\left(\delta_1\right) + A_2^2V\left(\delta_2\right) + 2A_1A_2Cov(\left(\delta_1, \delta_2\right)$$

$$If\ A3\ is\ violated\ Cov\left(\delta_1, \delta_2\right) \neq 0$$

Case I: A3 holds

$$V_R(\hat{\beta}_{2,LS}) = \frac{1}{\left[\sum_{n=1}^{N}(R(.)-\overline{R})^2\right]^2}\sum_{n=1}^{N}\left[\left(R\left(\vec{S}_n\right)-\overline{R}\right)^2 V\left(\vec{\delta}\left(\vec{S}_n\right)\right)\right] = \frac{1}{\left[\sum_{n=1}^{N}(R(.)-\overline{R})^2\right]^2}\sigma^2\sum_{n=1}^{N}(R(.)-\overline{R})^2$$

$$= \frac{\sigma^2}{\sum_{n=1}^{N}\left(R\left(\vec{S}_n\right)-\overline{R}\right)^2}$$

And then, you can show, of course, you will be able to show that V_R is equal to 1 over summation n equals 1 to N, R minus R bar the squared whole squared variance of. Now, R is a constant, so you know as we move, as we multiply R minus R bar, R minus R bar with beta 1, beta 2 R, we are working with constants. And the variance of constants is 0.

So, the only thing that is going to remain inside the variance operator is the random variable term delta. So, I am going to work with 1 to N, R minus R bar times delta S_n.

So, as in case 1, I am going to work with assumption A 3 holds that there is no covariance between the error term at different locations. So, now, if I were to focus, if you were to just you know bring your focus to this variance term, right of course, its variance is conditional on R. So, R is a constant. But now evaluate it a little bit closely.

So, I have written the variance of R with a constant. So, I am going to just replace this with a constant A, of course, A_n, but it is a constant. And this is by a random variable delta n just to sort of make the notation easier. So, summation n equals 1 to N, A_n delta n. This variance if capital N were just 2, you would write the following, you would write this as A_n squared variance of delta sorry; A_1 squared, right?

So, I am going to work with it for a case when N equals 2, this will be the A_1 squared variance of delta 1 plus A_2 squared variance of delta 2 plus 2 A_1 A_2, the covariance of delta 1 and delta 2, right? This is for the case when we had N, capital N equal 2, right? We are very well aware of how the variance operator works for that simplistic case.

Now, extend that understanding to the general case when we have more than 2 variables, random variables being summed inside a variance operator. So, my random variables are delta S_1, delta S_2, delta S_3, all the way 3 through delta S_n. So, this variance in the general form, when we have homoscedastic or spherical errors, all the covariance terms are going to vanish to 0. So, all I am going to remain with is the term with the variance.

And standing outside is the coefficient vector which is squared. So, I am going to use this to write down the variance of beta 2 hat LS given R you know when A 3 holds. So, I am going to do that now. So, when A 3 holds, the variance of conditional variance of beta 2 hat LS is going to be 1 over summation n equals 1 to capital N, R minus R bar the whole squared and the whole thing squared times summation; I am going to apply my understanding of how the variance operator works when the covariance terms are 0.

So, n equals 1 to N, R S_n minus R bar squared, and then apply the variance operator on each of the delta and S_n, or the delta S_n. And this variance of delta S_n, this value is nothing, but equal to sigma square. It is a constant.

So, the constant will directly come out of the summation term. So, I have 1 over summation n equals 1 to N, R minus R bar squared whole thing squared, times sigma square times n equals 1 to N, R minus R bar the whole thing squared.

And hence, I have, I can cancel one of the denominators, you know terms, with what is sitting in the numerator about summation R minus R bar squared. So, that is why the variance of beta 2 hat o LS will come out to be sigma square divided by summation n equals 1 to N, R minus R bar the whole squared.

Now, where now it starts to appear, if A 3 were to be violated or if A 3 does not hold, or it fails where will the difference apply, well the difference will apply in the case that these covariance terms will no longer be 0, when if A 3 is violated. So, that is something we are going to look at as the next case.

$Case\ II\ when\ A3\ fails$

$Given\ variance\ -\ covariance\ matrix\ of\ model\ errors;\ \left(Cov(\delta\left(\vec{u}\right), \delta\left(\vec{v}\right)\right) \equiv \Omega_{NXN}$

$\vec{u} \in \left\{\vec{\delta}_1, \ldots, \vec{\delta}_n\right\}$

$\vec{v} \in \left\{\vec{\delta}_1, \ldots, \vec{\delta}_n\right\}$

$\Omega\ NXN\ = \left[\sigma^2\ \sigma^2\rho\ \sigma^2\rho^2\ \ldots\ \sigma^2\rho^{n-1}\ \sigma^2\rho\ \sigma^2\ \sigma^2\rho\ \ldots\ \sigma^2\rho^{n-2}\ \vdots\ \vdots\ \sigma^2\rho^{N-1}\ \vdots\ \vdots\ \sigma^2\rho^{N-2}\ \ddots\ \vdots\ \ldots\ \ldots\ \ddots\ \sigma\right.$

$$V_R(\hat{\beta}_{2,LS}) = \frac{1}{\left[\sum_{n=1}^{N} (R(.)-\bar{R})^2\right]^2} V_R\left(\sum_{n=1}^{N} (R(.) - \bar{R}).\ \delta(\delta_n)\right)$$

$$= \frac{1}{\left[\sum_{n=1}^{N} (R(.)-\bar{R})^2\right]^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \left(R\left(\vec{S}_n\right) - \bar{R}\right)\left(R\left(\vec{S}_m\right) - \bar{R}\right) Cov\left(\delta\left(\vec{\delta}_n\right), \delta\left(\vec{\delta}_m\right)\right)$$

$$\neq \frac{\sigma^2}{\sum_{n=1}^{N} (R(.)-\bar{R})^2}$$

$No\ longer\ minmum\ variance\ in\ case\ of\ spatial\ dependence\ in\ model\ errors.$

So, let us say case 2 when A 3 fails, right? And we are talking about a very specific structure that we have articulated, you know in the covariance, variance-covariance matrix. That is to say that with the given structure, you know given the variance-covariance matrix of model errors that is to say that covariance of delta u delta V, right?

And let us say if we were to evaluate it for all the locations in u being S_1 till S_n and V being locations S_1 till S_n, right? So, these are variables, we know that we can write this, we can define this as n by n matrix because you will have n factors of the rho where u is varying given a value of V. And you know for each given value of u or location of u we can vary the V locations from S_1 to S_n, right?

This omega matrix for the given structure will be written as follows when u equals V. We have the covariance given as, let us write it down that we have this as sigma squared rho u minus V. So when u equals V which is a diagonal element, I have sigma squared, sigma squared all the way sigma squared. When I move from location 1, S_1, when I will move from location S_1 to location S_2, I have my variance changes to sigma squared rho. When I move on to S_3, my covariance will become sigma squared rho squared, all the way till sigma squared rho N minus 1.

Similarly, when I am looking at the second S_2, location S_2, in the row and I move across columns S_1 will have sigma squared row that is u minus V is just 1. Here I have an S_3 also I will have sigma squared rho because 2 minus 3 is also equal to 1 and till sigma square rho N minus 2. So, I can keep going. The first term here will be sigma square rho N minus 1 which is the difference between capital N and location 1, and I have capital Ns everywhere. So, I hope this is clear sigma square rho N minus 1, keep going sigma square rho and so on and so forth, right? So, now, the off-diagonal elements are nonzero. This is a specific form of heteroscedasticity that we have introduced through spatial dependence.

So, now, if the covariance terms are nonzero, let us specify the variance of beta 2 hat again. So, we are going to specify this for this case, right? So, I am going to just write it down again I have a variance of beta 2 hat least squares is given as 1 over summation n equals 1 to N, R minus R bar the whole squared, the whole thing squared. And then the variance of R summation n equals 1 to N, R minus R bar times delta S_n.

I am going to write this as follows. I am going to generalize it, You have seen this form before in one of the earlier lectures. I am going to write down this as summation n equals 1 to N and summation m equals 1 to N, right? So, I am moving u's and V's one by one.

I am going to write it as R Sn minus R bar R S_m minus R bar times the covariance of delta Sn comma delta Sm. So, this is the most general form of the variance, the variance of beta 2 hat o LS. And very clearly you will start, of course, with these covariance terms, we will be directly sourced from the omega matrix which is my n by n matrix of the variance and covariance factors, right?

But what happens is that this is no longer equal to the minimum variance form which is n equals 1 to N, R minus R bar the whole square. So, we no longer have minimum variance in case of spatial dependence in model errors, right? So, the final implication is that the least squares estimators are not the best, right? So, they are no longer best under spatial dependence. So, this is the implication that I no longer have a very precise estimator. So, least squares are still unbiased, but it is not so precise, right?

So, this is the implication that we have drawn from when we introduce spatial dependence to the regression model, right?

$Hence,\ LS\ estimators\ are\ not\ efficient\ in\ the\ presence\ of$

$spatial\ dependence\ in\ moedel\ errors.$

$\Longrightarrow LS\ are\ NOT\ B\ L\ U\ E$

$Efficient\ estimators\ under\ heteroskedadtic\ errors$:

$Generalized\ Least\ Squares\ estimators\ or\ GLS\ estimators.$

So, as a next step, right, first we are going to just say that hence we are going to state formally that least squares estimators are not efficient, not efficient in the presence of spatial dependence in model errors, right? And now this spatial dependence in model errors as we have seen, as have stated earlier is a form of heteroscedasticity, right? So, that means, that LS are not blue, right?

And remember, they are still linear, they are still unbiased, they are still an estimator, but the trouble is they are no longer best. That is why they are not you know a blue estimator. Now, what I am going to do next is I am going to state that the efficient estimators under heteroscedastic errors in general are called the generalized least squares estimators or better known as the GLS estimators.

Now, given, so basically we have moved from a specialized homoscedastic variance-covariance structure to a more general variance-covariance structure where the off-diagonal elements are not 0. And as a consequence, the efficiency property is taken away from the least squares, the ordinary least squares estimator. But they are then restored if we work with what is called the generalized least squares estimators.

So, in the next part of this lecture, what I am going to do is I am going to formally introduce the GLS estimators. And then, I am going to provide you with a general strategy for estimating such estimators, to getting such coefficient estimators when you have spatial dependence of a general form in the data, alright.

So, see you in the next part of the lecture.

Thank you for your attention.