Health Economics

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Week – 10

Lecture 50- Data Envelopment Analysis: BCC Model

Welcome, friends, once again to our NPTEL MOOC module on Health economics; we have been explaining the chapter on health efficiency or the week on health efficiency. The specific chapter is in continuation to the DEA model that is Data Envelopment Analysis. The specific goals of this particular lecture which is in continuation to the previous one, we are going to clarify further on the output oriented CCR model that is the CRS model and we are also emphasizing once again on the BCC model. We discussed just the basics, but now we are clarifying the BCC model, its respective derivation, and its particular efficiency scores, etcetera. We will also clarify the conditions for efficiency scores and the efficient targets.

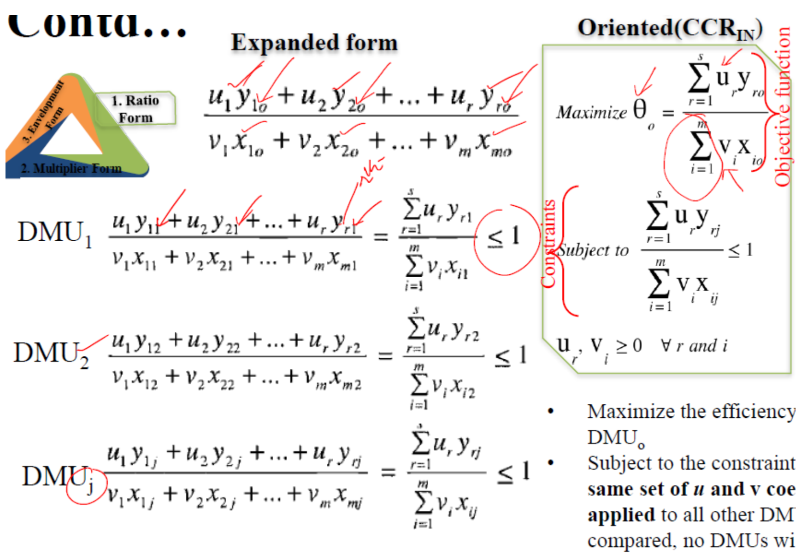
So, to start with, this is the one we have said already; there are three important approaches. Basically, in the previous lecture, we discussed the formulation and estimation that is based on three forms of input-oriented CCR model. This is largely explained. Here, we will be emphasizing the output-oriented CCR model.

As to explain in short starting with the ratio form, then we will discuss a bit of multiplier form and then we will discuss about DEA, envelopment form. So, in the input-oriented model, the CCR input model, especially that is based on the principles of constant returns to scale, the objective function is to maximize, subject to maximize the output with respect to the inputs and their respective weights we already emphasized in our earlier lectures and this is indeed our objective function and in the input-oriented model, we take it as θ and subject to the constraint is here where we say that the other DMUs and their output to inputs should be actually less than equal to 1. Only then can we discuss the maximization of the concerned DMUs and their respective output. The weights that were just discussed should be actually positive for all the rth levels of output and ith levels of inputs. So, now maximizing the efficiency rating θ for DMUs O, O is the Oth DMUs we are discussing.

So, we have already discussed our constraint function as well as the constraint function where the U and V coefficients are also taken, and their respective weights are also multiplied with the output and inputs, respectively. No DMUs will be more than an efficiency score of 1. That is what we have already explained. Though we have talked about this in the previous lecture, the expanded form was not discussed. Here is the expanded form of how it looks exactly. So, since we said summation of U_r and Y_{ro} for the Oth DMUs and rth output of its, you can just see how the Oth output is actually or the Oth not output, Oth DMU is actually mentioned everywhere and for their respective weights and their outputs are also discussed.

Here is the output. Similarly, it is actually since we are emphasizing on the ratio form, so we are taking as a ratio to their respective inputs. And inputs and their respective weights are multiplied. So, as a summation, since summation, we have already considered it in our formula and as per the original suggestions. So, these are all.

You can further clarify your concepts for DMU 1 and DMU 2, similar to DMU J, as far as the constraint function is concerned. Each DMU and its objective function should be less than equal to 1. You can just have a look. This is for the first DMU, everywhere it is 1 and for rth level of output everywhere we are actually concerning with the rth level of output. So, this is till the r, r you can just see, this is written as rth for the respective DMU that is at this moment we are explaining 1 and it has to be ensured that these summations should be less than 1 as a ratio.



Similarly, for DMU 2 to DMU j, this is how it looks like, and it is calculated in a disaggregated format. And we are sticking to the input-oriented model and in short it is also called CCR_{IN}. So, in terms of the output-oriented model, it is called CCR_{OUT}.

$$\text{Min } \textstyle\sum_i v_i x_{io} / \sum_r u_r y_{ro}$$
$$\text{Subject to}$$
$$\textstyle\sum_i v_i x_{ij} / \sum_r u_r y_{rj} \geq 1 \text{ for } j = 1, \ldots, n,$$
$$u_r, v_i \geq \varepsilon > 0 \text{ for all } i \text{ and } r.$$

In that case, the approach is to minimize the cost with respect to output, and that is what we usually take. In that case, we take another symbol. We will just discuss, but in the input-oriented model, we have taken as θ, and here the minimization of input with respect to the output subject to its constraints for the jth DMUs and I will also clarify what is this Vi and Ur and Vr that is precisely the weights, but the exact clarification is called they are, their limits would be very very at a very smaller number, and that is called Archimedean element noted by epsilon e, it is actually noted as very small element, and that should be higher than 0, and that U and V should be not very high, it should be very less and it should be positive real number. And these are all we have also partly discussed earlier, but further clarification was just given. In terms of another form called multiplier form, how input-oriented model looks like and we explained earlier that it is multiplied with the respective way to its output of the concerned DMU that is Oth DMU, and our objective function is mentioned as θ subject to the difference between the output of the jth DMUs to the jth inputs, jth DMUs, and their respective inputs.

The summation is also made, and the constraint function is clearly clarified that the difference should also be less than equal to 0, and it is attached with another constraint function, especially on the inputs of the Oth that is the DMUs unit and its input should be the sum of the inputs should be 1 and we have already clarified that the Ur and Vi should be positive. Actually, this multiplier method in the CCR form or the input-oriented method can also be written as maximize Z with respect to μ; μ is also written, and the rest are more or less the same, and μ is also interpreted. The transformation developed by, transformation developed by Charnes and Cooper in 1962 for linear fractional programming selects a representative solution, and that is, the solution is U and V, for which the product of Vi Xi for the inputs of the Oth DMU should be equal to 1. This yields the equivalent linear programming problem in which the change of variables from U and V to μ and V is a result and mentioned by Charnes Cooper transformation. That is why it is called a transformation.

Input Oriented(CCR_{IN})

Maximize $\theta_o = \sum_{r=1} u_r y_{ro}$

Subject to :

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0$$

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$u_r, v_i \geq 0$$

Also be written as

$$\max z = \sum_{r=1}^{s} \mu_r y_{ro}$$

subject to

$$\sum_{r=1}^{s} \mu_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0$$

$$\sum_{i=1}^{m} v_i x_{io} = 1$$

$$\mu_r, v_i \geq 0$$

Output Oriented(CCR_{out})

$$\min q = \sum_{i=1}^{m} v_i x_{io}$$

DMU

subject to

$$\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} \mu_r y_{rj} \geq 0$$

$$\sum_{r=1}^{s} \mu_r y_{ro} = 1$$

$$\mu_r, v_i \geq \varepsilon, \quad \forall r, i$$

In output-oriented format, that is called CCR output. This is just the reverse, and it is simply mentioned as a minimum of Q. G we take in maximization of output subject to its constraints, but here it is just the reverse as minimization of inputs of the Oth DMUs. So again, if you are following very correctly, this is the concerned DMU unit, and even the

Archimedes E is also written, and its value we have defined already in the previous slide, and now μ is a transformed format. The solution remains the same in ratio and even in multiplier form.

That is another note you can also follow. The most awaited one with our title of the lecture is on envelopment form, and its expanded format will explain, and its full form will also explain. The θ, as I already mentioned, is an input-oriented one, whereas, in the output one, we use Θ. Θ* that is to minimize the θ subject to the constraints. Here, the constraints are already explained in an earlier lecture, but the respective potential efficiency score, which we derived from the envelopment form, has to be a product of its first constraint in the inputs.

Input Oriented (CCR_{IN})

$$\theta^* = \min \theta$$
subject to
$$\sum_{j=1}^{n} x_{ij}\lambda_j \le \theta x_{io} \quad i = 1,2,\ldots,m;$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j \ge y_{ro} \quad r = 1,2,\ldots,s;$$
$$\lambda_j \ge 0 \quad j = 1,2,\ldots,n.$$

$$\min \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$
subject to
$$\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = \theta x_{io} \quad i = 1,2,\ldots,m;$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = y_{ro} \quad r = 1,2,\ldots,s;$$
$$\lambda_j, s_i^-, s_r^+ \ge 0 \;\forall i, j, r$$

Output Oriented (CCR_{out})

$$\max \phi$$
subject to
$$\sum_{j=1}^{n} x_{ij}\lambda_j = x_{io} \quad i = 1,2,\ldots,m;$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j = \phi y_{ro} \quad r = 1,2,\ldots,s;$$
$$\lambda_j \ge 0 \quad j = 1,2,\ldots,n.$$

$$\max \phi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$
subject to
$$\sum_{j=1}^{n} x_{ij}\lambda_j + s_i^- = x_{io} \quad i = 1,2,\ldots,m;$$
$$\sum_{j=1}^{n} y_{rj}\lambda_j - s_r^+ = \phi y_{ro} \quad r = 1,2,\ldots,s;$$
$$\lambda_j \ge 0 \quad j = 1,2,\ldots,n.$$

And then the λ coefficients associated with the quantity of output and input utilizes production. Another one is on output aspects. So, we can just check the difference very clearly. This is where we say maximization of output; that is, the symbol is taken as Θ subject to again; in this case, the constraints function is deriving the potential efficiency score of Θ, and that is part of the constraints function. λ is the respective coefficient associated with the quantity of input-output, etcetera, which is also considered.

The full model is written like this, where we are also encapsulating or understanding the slacks, the slacks we have already input slack, and the output slacks we have already discussed. In the full model, we are not just directing the minimization of inputs to output or maximization of output to input, etc. So, here the slacks are respectively mentioned, and even in the subject constraints function, we have included their respective slacks and their coverage is mentioned. Similarly, in the output-oriented model, the slacks are also discussed. You can just see how, of course, the slacks in the input case, we are trying to minimize the input where in the, therefore, it is minus where in the output-oriented model the slacks are actually additionally improving your extent of output.

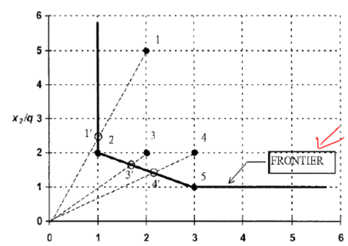
The rest are also taken care of in the slacks and in the constraint function. We are referring to the work of Afriat 1972, Fare, Grosskopf and, Logan and Banker etcetera. They suggested adjusting the CRS-DEA model. Hence, this is presented here. When economies of scale exist, what will happen? What do you mean by this? Basically, this suggests if a proportional increase in one or more inputs can cause greater than proportional change or increase in output.

Similarly, a proportional increase in inputs may yield a less-than-proportional increase in output. So, in the CRS input-oriented DEA framework, we have taken 5 firms. As I already said, 2 are inputs and a single output. So, the output is here Q, and the DMU is 5; here are DMUs. We will stick to a specific DMU and clarify how it works in software. In our next lecture, we will give you the direct application, how you guys can derive results, and what the interpretation should be.

Example

CRS input-orientated DEA (five firms that use **two inputs** to **produce a single output**)

Firm	Q (output)	x_1	x_2	x_1/q	x_2/q
1	1	2	5	2	5
2	2	2	4	1	2
3	3	6	6	2	2
4	1	3	2	3	2
5	2	6	2	3	1



CRS Input-Orientated DEA Example

So, here, for a radial measure, we are taking with respect to their respective output of the inputs. Hence, those can be measured X1 upon Q and X2 upon Q. so, the input-output ratio for this example is plotted in this picture or in this graph. This DEA frontier, which is derived and highlighted in bold line and written as a frontier in our graph, is indeed the result of running 5 linear programming problems. So, there are 5 linear programming problems we will just clarify.

For firm 3, we are now sticking to firm 3 and how the efficiency is calculated from the distance we already explained. Here, in firm 3, the target is to minimize Q with respect to the constraints. That is for the other one, subject to the constraint is that here it is minus Q3 plus Q1λ1 and Q2λ2, then Q3λ3 and so on that has to be greater than is equal to 0 because that is the constraint function we already discussed. For the other two constraint functions, I also mentioned θ times X13, and then the constraint has to be subtracted from the θ X13, then θ X23 for all λ to be positive. So, you can just have a look.

- for firm 3 equation as

slide 7 {Input Oriented (CCR_{IN})}

$min_{\theta,\lambda}\Theta$

St $x_{11}\lambda_1 + x_{12}\lambda_2 + x_{13}\lambda_3 + x_{14}\lambda_4 + x_{15}\lambda_5 \geq \Theta x_{13}$

$x_{21}\lambda_1 + x_{22}\lambda_2 + x_{23}\lambda_3 + x_{24}\lambda_4 + x_{25}\lambda_5 \geq \Theta x_{23}$

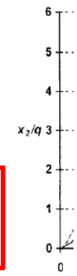
$y_{11}\lambda_1 + y_{12}\lambda_2 + y_{13}\lambda_3 + y_{14}\lambda_4 + y_{15}\lambda_5 \leq y_{13}$

$\lambda \geq 0$

$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$

Can also be written as

$$-q_3 + (q_1\lambda_1 + q_2\lambda_2 + q_3\lambda_3 + q_4\lambda_4 + q_5\lambda_5) \geq 0,$$
$$\theta x_{13} - (x_{11}\lambda_1 + x_{12}\lambda_2 + x_{13}\lambda_3 + x_{14}\lambda_4 + x_{15}\lambda_5) \geq 0,$$
$$\theta x_{23} - (x_{21}\lambda_1 + x_{22}\lambda_2 + x_{23}\lambda_3 + x_{24}\lambda_4 + x_{25}\lambda_5) \geq 0,$$



The values of θ and λ, which provide a minimum value for θ are listed below

Input Slack & Output Slack

CRS Input-Orientated DEA Results

firm	θ	λ_1	λ_2	λ_3	λ_4	λ_5	IS_1	IS_2	OS
1	0.5	-	0.5	-	-	-	-	0.5	-
2	1.0	-	1.0	-	-	-	-	-	-
3	0.833	-	1.0	-	-	0.5	-	-	-
4	0.714	-	0.214	-	-	0.286	-	-	-
5	1.0	-	-	-	-	1.0	-	-	-

The values of θ and λ, which provide a minimum value for θ, are discussed here. So, in the figure, we have tried to give the directions for IS here. IS stands for the input slacks, and OS stands for the output slacks for all 5 DMUs. So, you can just see, especially for firm 3, which you have already started explaining, In firm 3, the figure is here (above), and the θ value is 0.833. We are just presenting it here. Firm 3 could possibly reduce the consumption of all inputs by 1 minus this θ, which is around 16.7 percent, without reducing its output. This implies production at the point denoted 3 prime; you can just have a check that 3 prime is here. This projected point 3 prime lies on a line joining 0.2 and 0.5, this straight line. Therefore, 2 and 5 are usually referred to as the peers of firm 3. They define where the relevant part of the frontier is, and hence, they define efficient production for the firm. 3. The point 3 prime is basically a linear combination of points 2 and 5 where the weights in this linear combination are with the λs, which we have already discussed. In row 3 of the above table, we have already discussed this λ, this row 3, and their respective λ.

And then these are, I think, the clarifications related to the CCR model, which is very basic so far as efficiency scores based on CCR are concerned, and that is based on the constant returns to scale assumption. Hence, that does not capture the specific efficiency. Indeed, the constant returns to scale capture the general efficiency score or total efficiency score, which is divided into either pure efficiency score or pure efficiency and scale efficiency. Sometimes, pure efficiency is also called managerial efficiency, and we will also clarify. When we say pure efficiency, we are actually referring to the production function, which is, in reality, pure or, in reality, following variable returns to scale.

Hence, the returns to scale models are referred to in the context of BCC. BCC, we have already clarified this in our previous two lectures. The CRS (constant returns to scale) assumption is appropriate when all firms are operating at an optimal scale. Due to imperfect competition, government regulations, constraints on finance, etc., firms are not operating at optimal scale. So, again, referring these papers for further details.

CRS, we suggested adjusting the CRS-DEA model. I think we referred to it already, and we also discussed this part in the previous slide. The CRS linear programming problem is easily extended to account by the VRS, and the variable returns to scale by adding to the convexity constraint to CCR. So, given the CCR as a constant linear function, a production function, we are addressing the convexity constraint in comparison to the CCR. So, we will clarify. Basically, it forms a convex hull or envelope data points that are more tightly tied to the CRS conical hull.

The CRS, if you remember it, is the upward sloping from the origin and the 45-degree line defining one optimum efficiency score. From there, we are going to define the production function, which is forming a convex hull. Or the point that is very tightly defined by the constant returns to scale, which is the linear conical hull. This provides technical efficiency scores that are greater than or equal to those obtained using the CRS model. Convexity constraints ensure that an inefficient firm is only benchmarked against firms of a similar size.

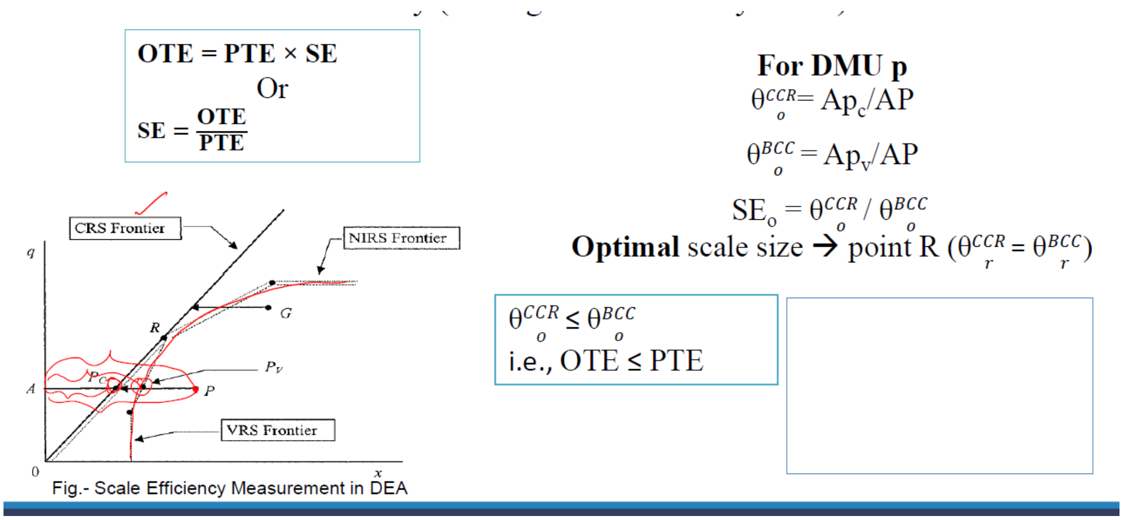
That is another important direction. This is where we are benchmarking. However, in the earlier case, we are not benchmarking to a similar size, which might be different in the CCR model. So, we are going to emphasize the convex combinations once again. In CCR, they did not scale up and scale down, including PPS and over-utilizing or under-utilizing. In CCR, I have already mentioned overall technical efficiencies.

Inefficiency occurs for two reasons. Firstly, we estimate inefficiency to identify the efficiencies. So, we are emphasizing managerial efficiency at this moment due to the inappropriate scale size used. Hence, BCC, which is pure technical efficiency or managerial efficiency, is defined as this: So, here we are referring to the BCC model. In CRS DEA, the firm may be benchmarked against firms that are substantially larger than it.

But in that case, in the BCC model, we are defining that should be equal to 1. That means, in the CRS DEA, the λ sum should be less than or greater than 1. But in this case, λ should be equal to 1 in the BCC model. So, the BCC is called the pure technical efficiency model or, managerial efficiency model or score. We are going to get the managerial efficiency score in this case.

So, the overall technical efficiency score is nothing but the pure technical scores times the scale efficiency, or scale efficiency can be derived with OTE divided by PTE. So, the figure that I already mentioned is based on the CRS frontier, and this is based on the production possibility, which is non-linear or different from this. It is actually based on the variable returns to scale principle. And so, this is also called the VRS frontier. And you can see for DMU's P, if you refer here DMU's P, so, the input-oriented model that is the θ CCR for the

Oth organization, which already discussed, it is here the Oth organization is we are actually referring to the P.



That is, in this case, the Oth DMU is mentioned. So, here, the θ is calculated as AP, APC, APC. You can just see it divided by AP. Meanwhile, in the BCC framework or principle, APV is divided by AP. For the BCC, you can just see this divided by AP. Since the Pth level of the frontier is mentioned as P and the optimum frontier size is respectively different, this is one in the CCR case, and this is in the BCC case.

So, the optimal scale size is defined at R because both BCC and CCR are equal. If the θ in the CCR model is less than equal to the θ value in the BCC model, in that case, the overall technical efficiency is actually lesser than that of the pure technical efficiency. So, in the BCC model, which we already discussed, the full format of the BCC model we already discussed including the slacks, and the target was to minimize the θ and the constraint functions are also respectively mentioned. And here you can see the sum of λ, which already said for the jth DMUs that should be equal to 1, and that is also called convexity constraint to form the convexity constraint that has to be equal to 1. So, BCC, that is, the BCC, is equal to CCR plus the convexity constraints.

BCC model (Mathemati

Input Oriented (CCR_{IN})

$$\text{Minimize } \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io} \quad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} \quad r = 1, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1 \quad j = 1, \ldots, n$$

$$\lambda_j \geq 0 \quad j = 1, \ldots, n$$

Efficient target

Inputs: $\hat{x}_{io} = \theta^* x_{io} - s_i^{-*} \quad i = 1, \ldots, m$

Outputs: $\hat{y}_{ro} = y_{ro} + s_i^{+*} \quad r = 1, \ldots, s$

Output Oriented (CCR_{out})

$$\text{Maximize } \phi + \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right)$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \quad i = 1, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \phi y_{ro} \quad i = 1, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j = 1 \quad j = 1, \ldots, n$$

$$\lambda_j \geq 0 \quad j = 1, \ldots, n$$

Efficient target

Inputs: $\hat{x}_{io} = x_{io} - s_i^{-*} \quad i = 1, \ldots, m$

Outputs: $\hat{y}_{ro} = \phi^* y_{ro} + s_i^{+*} \quad r = 1, \ldots, s$

And in the output-oriented case, that is to maximize the Θ plus all possible slacks (S). Then, the rest of the details will be discussed. Another addition to the problem is that there are some efficient targets we also discussed in the lecture, such as input to output, and their respective targets are discussed with their respective slacks.

In the output-oriented case, you can also see the target here, and the Θ is noted correctly. A firm may be benchmarked against firms that are substantially larger or smaller than it. The λ sum of their weights may be less than or greater than 1, which we already discussed, but in this case, the jth firms and their sum of λ should be equal to 1 in order to define the convexity constraints. Let us be the case that there are 5 firms and a single output using a single input, and considering input-oriented DEA, the data are listed in this table. And yes, this is the case, so it is a single input and single output case.

Firm	Q(output)	X (Input)
1	1	2
2	2	4
3	3	3
4	4	5
5	5	6

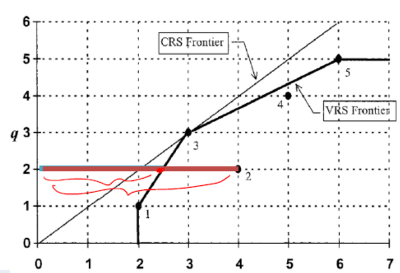


Fig.- VRS Input oriented DEA

Firm	CRS TE	VRS TE	Scale	
1	0.5	1	0.5	IRS
2	0.5 (2/4)	0.625 (2.4/4)	0.8 (0.5/0.625)	IRS
3	1	1	1	
4	0.8	0.9	0.889	DRS
5	0.833	1	0.833	DRS
mean	**0.727**	**0.905**	**0.804**	

We are just trying to calculate the CRS-based technical efficiency and VRS-based technical efficiency score. In CRS based simply it is if suppose in 2th and in second firm, it will be equal to you can see 2 it will be output by input, so 2 by 4, 1 by 2, 2 by 4 it is like this, 1 by 2 and 2 by 4, 3 by 3 etcetera for CRS based estimation. Whereas in VRS-based technical efficiency, you can see for the second firm, I will just clarify it will be 2.4 around this figure per second divided by the entire 4, this till this. So, this is the answer, and for the scale estimation, you can see it is basically 0.5 divided by the CRS technical efficiency divided by VRS technical efficiency, which we already calculated, and hence, it is 0.8. If it is actually 0.8, it signals an increase in returns to scale. This is how we highlight and mention things in the diagram.

This is the one for the VRS case. That has to be the first one is actually as simple as that 1 by 2. So, the scale efficiency we have already discussed, and I have also clarified how this works for the VRS, is very important. CRS you can easily calculate. Here it is this, this divided by the entire. Similarly, for the third, it will be this distance divided by this distance, the same distance, and then you can respectively calculate the others.

So, there are different healthcare perspectives that are important in this context. In the healthcare sector, imperfect competition constraints on finance, regulatory constraints on entry, merger, etc., result in organizations operating systems being inefficient and operating on an inefficient scale. Hence, the choice of CRS or VRS is an important decision-making process, and this relies on an analyst's understanding of the market constraints facing firms within a particular sector. If the CRS technology is inappropriately applied, then all hospitals will operate at a suboptimal scale, and then the estimates of the technical efficiency will be confounded by the effects of scale efficiency.

We need to note here that one shortcoming of measuring scale efficiency is that the value does not indicate whether the firm is operating in an area of increasing or decreasing returns to scale. This can be determined by running an additional DEA problem with non-increasing returns to scale, that is, NIRS as well. So, this aspect will also be addressed in our next lecture.

I think this NIRS, etc., is also mentioned here in this diagram. You can just have a look. NIRS frontier we have given, we will try to also clarify, and the rest of the details are in slides. I hope you have received a number of directions. You must have been charged enough to collect your data to measure efficiency scores and differentiate this CCR and BCC model. So, these are important suggestions. I think the next lecture will be directly from the practical side, and we will estimate it through the package.

We will use that package or the software called DEAP software. Thank you. See you in the next class.